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Scales of Banach Spaces, Theory of Interpolation and their Applications

Łukasz Dawidowski

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Scales of Banach Spaces, Theory of Interpolation and their Applications

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Preface

The task of this book is to present the theory of the scales of Banach spaces and the role they play in the modern theory of Partial Differential Equations. Some parts of the theory of interpolation are analysed here too. The book gathers the results of previous investigations on this subject, completed with the new ones.

The present study is directed to the mathematics students finishing already their university career, mathematicians and other people interested in mathematical science. To understand it, the basic knowledge from the fields such as a course on functional analysis containing the basics of Sobolev spaces and integral calculus in Banach spaces are required. Every reader should also be familiar with the theory of distributions and the Fourier transform, but the elementary theorems related to both of them can be found in Appendix A.

The book is divided into three parts. The first one introduces the reader into the theory of the interpolation spaces, gives its brief description and presents the basic properties of interpolation spaces.

As the precursors of the theory of interpolation we can consider M. Riesz and O. Thorin, who in the thirties proved the theorem of the interpolation of the spaces $L^p(\Omega)$. In 1939 the generalization of these results was published by J. Marcinkiewicz. After the Second World War the theory was investigated by i.a. A. Zygmund, A.P. Calderón, E. Gagliardo and J.L. Lions along with J. Petree, who perceived the interpolation spaces as traces for variants of Sobolev spaces. The more complete description of the theory of interpolation spaces can be found i.a. in the monographs of H. Triebel, L. Tartar and A. Lunardi.

The aim of the second, main part of the book, is to present the construction of the scale of the Banach spaces, generated by such an operator $A\colon D(A) \subseteq X \to X$ of the type (ω, M) that $0 \in \rho(A)$. In this part we can find the description which contains the characterization of the scale of Banach spaces and the scale of dual spaces.

The object of the last part is to show the applications of the theory given before in specific problems. We introduce the class of sectorial operators, which were widely considered i.a. by T. Kato, H. Tanabe and D. Henry, H. Amann, A. Lunardi. The connection of this class with the operators of the type (ω, M) is analysed here too. It is also possible to find in this part some properties of the scale of Banach spaces, defined for sectorial operators. Finally, we deal with the investigation of the existence and smoothness of the solutions of some Cauchy's problems, considered on different spaces on the scale of Banach spaces.

The structure of the book is as follows:

Chapter 1: introduces the definition of fractional powers of operator and describes their basic properties.

Chapter 2: first deals with the spaces D_p^σ and describes their properties and shows that they coincide with some interpolation spaces. Next, the classical approach to the theory of interpolation spaces is given and different methods of introducing these interpolation spaces, e.g. K-method and trace method of real interpolation and complex method of interpolation are presented. In this chapter can also be found the definition of the operator of the type $(\omega, M(\theta))$.

Chapter 3: this short chapter gives the characterization of the domains of fractional powers of operator through infinitesimal generators of bounded semi-groups or bounded analytic semi-groups.

Chapter 4: this crucial chapter contains two sections. The first one considers inductive limits and projective limits of sequences of Banach spaces and their properties. The second one shows construction of the scale of Banach spaces for the linear operator of the type $(\omega, M(\theta))$, which resolvent set contains 0. Next the characterization of the scale of Banach spaces is discussed.

Chapter 5: the aim of this chapter is to introduce a few examples of the scales of the Banach spaces. One of the examples leads to definition of the fractional Sobolev spaces, which are useful spaces considered in the theory of Partial Differential Equations.

Chapter 6: presents sectorial operators, describes their properties and gives some basic examples of such operators.

Chapter 7: is devoted to some applications of the theories given before. At the beginning we consider the operators defined on different levels of the scale, that is to say for the operator $A\colon D(A) \subseteq X \to X$ we consider its restrictions or extensions $A|_{X^z}$ for the spaces in the scale $(X^z)_{z\in\mathbb{R}}$, generated by this operator. We show that if the operator A is closed or sectorial then all the operators $A|_{X^z}$ are also closed or sectorial. Next we present the theorem which shows that under certain assumptions the scales of the Banach spaces can be achieved by using the method of complex interpolation spaces. Finally we will give the examples that justify the consideration of the spaces with fractional exponents on the scale.

Chapter 8: deals with the Cauchy's problem

$$\begin{cases} u_t + Au = F(u), & t > 0, \\ u(0) = u_0. \end{cases}$$

We consider the local X^z-solutions as well as their existence and uniqueness. Next we reveal a few examples of Cauchy's problems for which we search the local X^z-solutions.

Appendix A: contains basic facts referring to the theory of distributions and Fourier transform.

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Lastly, but above all, I thank my wife for her patience, support and help.

Chapter 1

Fractional powers of operators

In this chapter we construct a family of closed operators with domains which set up a special collection of subspaces of a given Banach space X. Results concerning fractional powers of operators can be found among others in monograph of H. Amann [4], H. Triebel [40], in a sequence of papers of H. Komatsu ([23], [24]), in books of J.W. Cholewa and T. Dłotko [9], R. Czaja [12].

Let $(X, \|\cdot\|)$ denote a Banach space and let us assume that $A\colon X \supseteq D(A) \to X$ is a closed linear operator with the domain $D(A)$ dense in X such that the resolvent set of A contains $(-\infty, 0)$ and satisfies

$$\|\lambda(\lambda - A)^{-1}\| \leq M, \quad \lambda \in (-\infty, 0), \tag{1.1}$$

where M is a constant independent of λ. When $0 \in \rho(A)$ then we call such type of operators ***positive operators of type M***. Because of the equality $A(\lambda - A)^{-1} = \lambda(\lambda - A)^{-1} - I$ there exists a constant L such that

$$\|A(\lambda - A)^{-1}\| \leq L, \quad \lambda \in (-\infty, 0), \tag{1.2}$$

When $0 \in \rho(A)$ we can define *fractional powers* of operator A for exponent $z \in \mathbb{C}$, $\mathrm{Re} z < 0$ by the Dunford integral.

Definition 1.0.1. For $z \in \mathbb{C}$, $\mathrm{Re} z < 0$, we define *fractional powers* of operator A by

$$A^z = -\frac{1}{2\pi i} \int_\Gamma \lambda^z (\lambda - A)^{-1}\, d\lambda, \tag{1.3}$$

where the path Γ encircles the spectrum $\sigma(A)$ counterclockwise, avoiding the negative real axis. Furthermore, we set

$$A^0 = I,$$

where I denotes the identity operator on X.

The set Γ can be chosen as the curve consisting of the three segments:

$$\begin{aligned}&\left\{-se^{-i\theta}\colon s\in\left(-\infty,-\frac{1}{4M}\right)\right\},\left\{\frac{1}{4M}e^{i\psi}\colon |\psi|\leq\theta\right\},\\&\left\{se^{i\theta}\colon s\in\left(\frac{1}{4M},\infty\right)\right\},\end{aligned}\tag{1.4}$$

with $\theta\in[\pi-\arcsin\frac{1}{2M},\pi)$. Now we show that the definition of A^z for $\mathrm{Re}z<0$ does not depend on the choice of a particular θ.

Proof. To prove it, let us consider $\pi-\arcsin\frac{1}{2M}\leq\theta_1<\theta_2<\pi$ and for a fixed $z\in\mathbb{C}$ such that $\mathrm{Re}z<0$ set

$$A^z_{\theta_1}=-\frac{1}{2\pi i}\int_{\Gamma_{\theta_1}}\lambda^z(\lambda-A)^{-1}\,d\lambda,\quad A^z_{\theta_2}=-\frac{1}{2\pi i}\int_{\Gamma_{\theta_2}}\lambda^z(\lambda-A)^{-1}\,d\lambda$$

where Γ_{θ_1} and Γ_{θ_2} are the curve Γ as in (1.4) with θ replaced, respectively, by θ_1 and θ_2. For $s\geq\frac{1}{2M}$ consider two arcs given by

$$C^s_1=\{se^{i\psi}\colon\theta_1\leq\psi\leq\theta_2\},\quad C^s_2=\{se^{i\psi}\colon\ -\theta_2\leq\psi\leq-\theta_1\}.$$

Observe that

$$\left\|-\frac{1}{2\pi i}\int_{C^s_1}\lambda^z(\lambda-A)^{-1}\,d\lambda\right\|\leq\frac{2M+1}{2\pi}\frac{s}{1+s}s^{\mathrm{Re}z}\int_{\theta_1}^{\theta_2}e^{-\psi\mathrm{Im}z}\,d\psi\xrightarrow[s\to\infty]{}0$$

and analogously

$$\left\|-\frac{1}{2\pi i}\int_{C^s_2}\lambda^z(\lambda-A)^{-1}\,d\lambda\right\|\xrightarrow[s\to\infty]{}0.$$

Therefore the Cauchy's Theorem (see Theorem 7.2.1 in E. Hille, [21], p.163) implies that for fixed $z\in\mathbb{C}$ with $\mathrm{Re}z<0$ the operator A^z does not depend on the particular choice of $\theta\in[\pi-\arcsin\frac{1}{2M},\pi)$. $\square$

We can prove (see Lemma 3.0.5 in R. Czaja [12]) that A^z is one-to-one operator for $\mathrm{Re}z<0$, thus we can define fractional operators A^z for $\mathrm{Re}z>0$. But first let us prove some easy lemmas which state that operators A^z for $\mathrm{Re}z<0$ are well-defined and convergent.

Lemma 1.0.2. *When $A\colon D(A) \subseteq X \to X$ is a positive operator of type M, then*

$$\Xi_M := \left\{ \lambda \in \mathbb{C} \colon \exists_{s \le 0}\ |\lambda - s| \le \frac{|s|}{2M} \right\} \subseteq \rho(A)$$

and

$$\|(\lambda - A)^{-1}\| \le \frac{2M+1}{|\lambda|}, \text{ for all } \lambda \in \Xi_M.$$

Proof. Fix $\lambda \in \Xi_M$. Then there exists $s \le 0$ such that $|\lambda - s| \le \frac{|s|}{2M}$. We have

$$\lambda - A = (s - A)(I + (\lambda - s)(s - A)^{-1}).$$

Letting $B = I + (\lambda - s)(s - A)^{-1} \in \mathcal{L}(X)$ we see that

$$\|I - B\| = |\lambda - s| \|(s - A)^{-1}\| \le \frac{|s|}{2M} \frac{M}{|s|} = \frac{1}{2} < 1.$$

It implies the existence of $B^{-1} \in \mathcal{L}(X)$ and

$$\|B^{-1} - I\| \le \frac{\|I - B\|}{1 - \|I - B\|} \le 1$$

Hence $\lambda \in \rho(A)$ and $(\lambda - A)^{-1} = B^{-1}(s - A)^{-1}$. Moreover, we have

$$\begin{aligned} \|(\lambda - A)^{-1}\| &= \|(B^{-1} - I)(s - A)^{-1} + (s - A)^{-1}\| \le \\ &\le \|B^{-1} - I\| \|(s - A)^{-1}\| + \|(s - A)^{-1}\| \le \\ &\le 2\|(s - A)^{-1}\| \le \frac{2M}{|s|} \le \frac{2M}{|\lambda|} \frac{|s| + |\lambda - s|}{|s|} \le \\ &\le \frac{2M}{|\lambda|}\left(1 + \frac{1}{2M}\right) = \frac{2M+1}{|\lambda|}, \end{aligned}$$

which completes the proof. □

It is easy to notice that

$$\left\{ \lambda \in \mathbb{C} \colon |\arg \lambda| \ge \pi - \arcsin \frac{1}{2M} \right\} \cup \left\{ \lambda \in \mathbb{C} \colon |\lambda| \le \frac{1}{2M} \right\} \subseteq \Xi_M,$$

which shows that we can indeed choose the integration path Γ in $\rho(A)$.

Lemma 1.0.3. *Let $A\colon D(A) \subseteq X \to X$ be a positive operator of type M and $z \in \mathbb{C}$ be such that* $\mathrm{Re} z < 0$. *Then the operator A^z is bounded and analytic in* $\{z \in \mathbb{C} \colon \mathrm{Re} z < 0\}$.

Proof. Since the integrand in (1.3) is a continuous function with values in $\mathcal{L}(X)$ and analytic of the variable $z \in \mathbb{C}$, the integral

$$B_n^z = -\frac{1}{2\pi i}\int_{\Gamma_n} \lambda^z(\lambda - A)^{-1}\, d\lambda,$$

where $\Gamma_n = \Gamma \cap \{\lambda \in \mathbb{C}\colon |\lambda| \le n\}$, $n \in \mathbb{N}$, is well-defined and constitutes an analytic function B_n with values in $\mathcal{L}(X)$ (see St. Saks, A. Zygmund [34], p.107). The integral in (1.3) is defined in an improper way. Below we show that it exists and is almost uniformly convergent in $\Pi_0 = \{z \in \mathbb{C}\colon \mathrm{Re} z < 0\}$. Let K be a compact subset of Π_0 and fix $\varepsilon > 0$. Then there exists $\delta_1 > 0$ and $\delta_2 < 0$ such that $|\mathrm{Im} z| \le \delta_1$ and $\mathrm{Re} z \le \delta_2 < 0$ for any $z \in K$. Let $n_0 \in \mathbb{N}$ be such that $n_0 \ge \frac{1}{2N}$ and

$$-\frac{2N+1}{\pi}\frac{e^{\theta\delta_1}}{\delta_2} n_0^{\delta_2} < \varepsilon$$

Having $n > m \ge n_0$ we see that

$$\begin{aligned}
\|B_n^z - B_m^z\| &= \Bigg\| \frac{1}{2\pi i}\int_{-n}^{-m} (-se^{-i\theta})^z(-se^{-i\theta} - A)^{-1}(-e^{-i\theta})\, ds + \\
&\quad - \frac{1}{2\pi i}\int_{m}^{n} (se^{i\theta})^z (se^{i\theta} - A)^{-1} e^{i\theta}\, ds \Bigg\| \le \\
&\le \frac{1}{2\pi}\int_{-n}^{-m} e^{\mathrm{Re} z \ln(-s) + \theta \mathrm{Im} z}\left(\frac{2M+1}{-s}\right) ds + \\
&\quad + \frac{1}{2\pi}\int_{m}^{n} e^{\mathrm{Re} z \ln s - \theta \mathrm{Im} z}\left(\frac{2M+1}{s}\right) ds \le \\
&\le 2\cdot\frac{1}{2\pi}\cdot(2M+1)e^{\theta|\mathrm{Im} z|}\int_{m}^{n} e^{\mathrm{Re} z \ln s}\frac{1}{s}\, ds = \\
&= \frac{2M+1}{\pi}e^{\theta|\mathrm{Im} z|}\int_m^n s^{\mathrm{Re} z}\frac{1}{s}\, ds = \\
&= -\frac{2M+1}{\pi}e^{\theta|\mathrm{Im} z|}\frac{1}{\mathrm{Re} z}\left(m^{\mathrm{Re} z} - n^{\mathrm{Re} z}\right) \le \\
&\le -\frac{2M+1}{\pi}e^{\theta\delta_1}\frac{1}{\delta_2} n_0^{\delta_2} < \varepsilon.
\end{aligned}$$

We conclude (see St. Saks, A. Zygmund [34], p.116) that $B \in \mathcal{L}(X)$ for $z \in \mathbb{C}$ such that $\mathrm{Re} z < 0$ and the mapping $z \mapsto B^z$ is analytic in $\{z \in \mathbb{C}\colon \mathrm{Re} z < 0\}$. □

Now we define fractional powers of operator A^z for $\mathrm{Re} z > 0$.

Definition 1.0.4. We set for $z \in \mathbb{C}$, $\mathrm{Re} z > 0$,

$$A^z = (A^{-z})^{-1}\colon X \supseteq R(A^{-z}) \to X.$$

Finally, we can prove that for $\mathrm{Re} z = 0$ an operator $B_z\colon D(A) \to X$, $B_z x = AA^{z-1}x = A^{z-1}Ax$, is a well-defined closable operator in X. Additionally, $\overline{B_0} = I$

Definition 1.0.5. Let $z \in \mathbb{C}$ be such that $\mathrm{Re} z = 0$. We define

$$A^z = \overline{B_z}\colon X \supseteq D(\overline{B_z}) \to X.$$

Within the following theorems we discuss some elementary properties of the fractional operators and a very important interpolation inequality, which will be useful in introducing a scale of Banach spaces. We omit proofs of them, which may be found among others in papers of R. Czaja [12], H. Komatsu [23], [24] or H. Tanabe [36].

Theorem 1.0.6 (Theorem 3.0.12 in [12]).

(a) *For $z \in \mathbb{C}$ such that* $\mathrm{Re} z \geq 0$, *A^z is a closed operator in X.*
(b) *If $z_1, z_2 \in \mathbb{C}$ are such that* $\mathrm{Re} z_1 > \mathrm{Re} z_2 > 0$, *then $D(A^{z_1}) \subseteq D(A^{z_2}) \subseteq X$.*
(c) *For $n \in \mathbb{N}$ the power A^n coincides with the product $\underbrace{A \cdots A}_{n}$.*
(d) *For $z \in \mathbb{C}$ such that* $\mathrm{Re} z \geq 0$ *we have* $\mathrm{cl} D(A^z) = X$.
(e) *If $z_1, z_2 \in \mathbb{C}$ are such that $\mathrm{Re} z_1, \mathrm{Re} z_2, \mathrm{Re}(z_1 + z_2) \neq 0$ or $z_1, z_2 \in \mathbb{R}$, then*

$$A^{z_1+z_2}x = A^{z_1}A^{z_2}x = A^{z_2}A^{z_1}x, \quad x \in D(A^w)$$

where $w \in \{z_1, z_2, z_1 + z_2\}$ and $\mathrm{Re} w = \max\{\mathrm{Re} z_1, \mathrm{Re} z_2, \mathrm{Re}(z_1 + z_2)\}$

Theorem 1.0.7 (Interpolation theorem, Remark 2.3.1 in [36]). *For any $\sigma < \tau < \theta$ there exists a constant $C(\sigma, \tau, \theta)$ such that*

$$\|A^\tau u\| \leq C(\sigma, \tau, \theta)\|A^\sigma u\|^{\frac{\theta-\tau}{\theta-\sigma}}\|A^\theta u\|^{\frac{\tau-\theta}{\theta-\sigma}}, \quad u \in D(A^\theta).$$

Next we show another characterization of the operator A^z.

Theorem 1.0.8 (Theorems 3.0.7 and 3.0.13 in [12]). *If* $n \in \mathbb{N}$ *and* $z \in \mathbb{C}$ *are such that* $0 < \mathrm{Re} z < n$, *then*

$$A^{-z} = \frac{\Gamma(n)}{\Gamma(n-z)\Gamma(z)} \int_0^\infty s^{-z+n-1}(s+A)^{-n}\, ds$$

and

$$A^z x = \frac{\Gamma(n)}{\Gamma(n-z)\Gamma(z)} \int_0^\infty s^{z-1}A^n(s+A)^{-n}x\, ds, \quad x \in D(A^n). \tag{1.5}$$

In this theorem operators A^z are independent of the chosen integer n. In the papers of H. Komatsu [23] we can find another definition of fractional powers of operator where we do not have to assume that A has a dense domain and $0 \in \rho(A)$. It was shown, however, that if $\mathrm{Re} z > 0$, A^z is an operator in $\overline{D(A)}$ and it is determined by a restriction A_D which has a dense domain in $\overline{D(A)}$. Thus our requirement on a domain $D(A)$ is not restrictive as far as we consider the exponent z with a positive real part. As a consequence, we have

$$(\lambda(\lambda + A)^{-1})^m x \xrightarrow[\lambda\to\infty]{} x, \quad m \in \mathbb{N}, x \in X. \tag{1.6}$$

Chapter 2

Interpolation spaces

M. Riesz proved the following "interpolation" result in 1926, in the case $p_\theta \le q_\theta$. This restriction was removed by O. Thorin in 1938. It was often called *the convexity theorem.*

Theorem 2.0.1. *If $p_0, p_1, q_0, q_1 \in [0, \infty]$ and a linear map A is continuous from $L^{p_0}(\Omega)$ into $L^{q_0}(\Omega')$ and from $L^{p_1}(\Omega)$ into $L^{q_1}(\Omega')$, then for $\theta \in (0,1)$ it is continuous from $L^{p_\theta}(\Omega)$ into $L^{q_\theta}(\Omega')$, where*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{2.1}$$

and one has

$$\|A\|_{\mathcal{L}(L^{p_\theta}(\Omega), L^{q_\theta}(\Omega'))} \le \|A\|^{1-\theta}_{\mathcal{L}(L^{p_0}(\Omega), L^{q_0}(\Omega'))} \cdot \|A\|^{\theta}_{\mathcal{L}(L^{p_1}(\Omega), L^{q_1}(\Omega'))}.$$

Thorin's proof used a property of the modulus of holomorphic functions, *the three lines theorem*, stating that if $f(z)$ is holomorphic in the strip $0 < \mathrm{Re} z < 1$, continuous on the closed strip $0 \le \mathrm{Re} z \le 1$ and such that $|f(iy)| \le M_0$ and $|f(1+iy)| \le M_1$ for all $y \in \mathbb{R}$, then one has $|f(\theta + iy)| \le M_0^{1-\theta} M_1^\theta$ for all $\theta \in (0,1)$ and $y \in \mathbb{R}$.

Later on the idea of Thorin was used again by E. M. Stein, a general *method of complex interpolation* was developed by A.P. Calderón, J.L. Lions and M. Krein.

If $f \in L^p(\Omega)$, then Hölder's inequality gives $\int_E |f|\,dx \le \|f\|_{L^p(\Omega)} \mu(F)^{1/p'}$ for all measurable subsets E of Ω (here μ denotes a measure which defines the space $L^p(\Omega)$ and p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$). J. Marcinkiewicz introduced a space sometimes called *weak L^p spaces* denoted by $L^{p,\infty}(\Omega)$, which is the space of (equivalence classes of) measurable functions g for which there exists a constant C such that

$$\int_E |g|\,dx \le C\mu(E)^{1/p'}$$

for all measurable subsets $E \subseteq \Omega$. It contains $L^p(\Omega)$, but if $\Omega \subseteq \mathbb{R}^N$ and $p \in [1, \infty)$, it also contains functions like $\frac{1}{|x|^{N/p}}$. In 1939, J. Marcinkiewicz published the following result, as a note without proof. Two different proofs were added later by M. Cotlar and A. Zygmund.

Theorem 2.0.2. *If $p_0, p_1, q_0, q_1 \in [0, \infty]$ and a linear map A is continuous from $L^{p_0}(\Omega)$ into $L^{q_0,\infty}(\Omega')$ and from $L^{p_1}(\Omega)$ into $L^{q_1,\infty}(\Omega')$, then for $\theta \in (0,1)$it is continuous from $L^{p_\theta}(\Omega)$ into $L^{q_\theta}(\Omega')$, under the condition that $p_\theta \leq q_\theta$, where p_θ and q_θ are given by* (2.1).

The results of M. Riesz, O. Thorin, J. Marcinkiewicz, were generalized as the *theory of interpolation.* The main contributors were N. Aronsztajn, A.P. Calderón, E. Gagliardo, J.L. Lions, J. Petree, but similar techniques have been also used by specialists of harmonic analysis, like E.M. Stein.

2.1 Spaces $\mathrm{D}_\mathrm{p}^\sigma$

By $L^p(X)$ we denote the space of all X-valued strongly measurable functions $f\colon (0, \infty) \to X$ such that

$$\|f\|_{L^p} = \left(\int_0^\infty \|f(\lambda)\|^p \, d\lambda/\lambda\right)^{\frac{1}{p}} < \infty, \quad \text{for } p \in [1, \infty);$$
$$\|f\|_{L^\infty} = \sup_{0<\lambda<\infty} \|f(\lambda)\| < \infty.$$

Here $d\lambda/\lambda$ denotes the Haar measure on multiplicative group $(0, \infty)$. Additionally we accept $p = \infty-$ as an index. $L^{\infty-}(X)$ represents the subspace of all functions $f \in L^\infty(X)$ such that $\lim_{\lambda\to 0} f(\lambda) = 0$ and $\lim_{\lambda\to\infty} f(\lambda) = 0$.

Definition 2.1.1. Let $0 < \sigma < m$, where σ is a real number and $m \in \mathbb{N}$ and $p \in [1, \infty]$. We denote by $D_{p,m}^\sigma = D_{p,m}^\sigma(A)$ the space of all $x \in X$ such that $\lambda^\sigma (A(\lambda + A)^{-1})^m x \in L^p(X)$ with the norm

$$\|x\|_{D_{p,m}^\sigma} = \|x\|_X + \|\lambda^\sigma (A(\lambda + A)^{-1})^m x\|_{L^p(X)}.$$

It is easy to see that $D_{p,m}^\sigma$ is a Banach space.

Proposition 2.1.2 (Proposition 1.2 in [24]). *If integers m and n are greater than σ, the spaces $D_{p,m}^\sigma$ and $D_{p,n}^\sigma$ are identical and have equivalent norms.*

Proof. It is enough to show that $D^\sigma_{p,m} = D^\sigma_{p,m+1}$ when $m > \sigma$. Because of (1.2) every $x \in D^\sigma_{p,m}$ belongs to $D^\sigma_{p,m+1}$. Since

$$\frac{d}{d\lambda}(\lambda^m(A(\lambda + A)^{-1})^m) = m\lambda^{m-1}(A(\lambda + A)^{-1})^{m+1},$$

we have

$$\lambda^\sigma(A(\lambda + A)^{-1})^m x = m\lambda^{\sigma-m}\int_0^\lambda \mu^{m-\sigma}\mu^\sigma(A(\mu + A)^{-1})^{m+1}x\ d\mu/\mu. \quad (2.2)$$

This shows

$$\|\lambda^\sigma(A(\lambda + A)^{-1})^m x\|_{L^p(X)} \leq \frac{m}{m-\sigma}\|\lambda^\sigma(A(\lambda + A)^{-1})^{m+1}x\|_{L^p(X)}.$$

□

Definition 2.1.3. Let $\sigma > 0$ and $p \in [1, \infty]$. We define D^σ_p as the space $D^\sigma_{p,m}$ with the least integer m greater than σ.

Remark 2.1.4 (Proposition 1.4 in [24]). If $\mu > 0$, $\mu(\mu + A)^{-1}$ maps D^σ_p continuously into $D^{\sigma+1}_p$. Futhermore, if $p \leq \infty-$, we have for every $x \in D^\sigma_p$

$$\mu(\mu + A)^{-1}x \to x \quad \text{as } \mu \to \infty$$

Proof. Let $x \in D^\sigma_p$. Since

$$\begin{aligned}&\|\lambda^{\sigma+1}(A(\lambda + A)^{-1})^{m+1}\mu(\mu + A)^{-1}x\| \leq\\ &\qquad \leq \mu\|\lambda(\lambda + A)^{-1}\|\|A(\mu + A)^{-1}\|\|\lambda^\sigma(A(\lambda + A)^{-1})^m x\| \leq\\ &\qquad \leq \mu ML\|\lambda^\sigma(A(\lambda + A)^{-1})^m x\|,\end{aligned}$$

$\mu(\mu + A)^{-1}x$ belongs to $D^{\sigma+1}_p$.
Let $p \leq \infty-$. If $x \in D(A)$, then

$$(A(\lambda+A)^{-1})^m\mu(\mu+A)^{-1}x = (A(\lambda+A)^{-1})^m x - (A(\lambda+A)^{-1})^m(\mu+A)^{-1}Ax$$

converges to $(A(\lambda + A)^{-1})^m x$ uniformly in λ. On the other hand, $(A(\lambda + A)^{-1})^m\mu(\mu + A)^{-1}$ is uniformly bounded. Thus it follows that $(A(\lambda + A)^{-1})^m\mu(\mu + A)^{-1}$ converges to $(A(\lambda + A)^{-1})^m x$ uniformly in λ for every $x \in X$. Since $\|(A(\lambda + A)^{-1})^m\mu(\mu + A)^{-1}\| \leq M\|(A(\lambda + A)^{-1})^m x\|$, we obtain the second part of our remark. □

Theorem 2.1.5 (Theorem 1.5 in [24]). *$D^\sigma_p \subseteq D^\tau_q$, if $\sigma > \tau$ or if $\sigma = \tau$ and $p \leq q$. The injection is continuous. If $q \leq \infty-$, D^σ_p is dense in D^τ_q.*

Proof. First we prove that D_p^σ, $p < \infty$ is continuously embedded in $D_{\infty-}^\sigma$. Let $x \in D_p^\sigma$. Applying Hölder's inequality to (2.2), we obtain

$$\|\lambda^\sigma(A(\lambda+A)^{-1})^m x\| \le \frac{m}{((m-\sigma)p')^{\frac{1}{p'}}}\|\mu^\sigma(A(\mu+A)^{-1})^{+1}mx\|_{L^p(X)},$$

where $p' = \frac{p}{p-1}$. Hence $x \in D_\infty^\sigma$. Considering the integral over the interval (μ, λ), we have similarly

$$\|\lambda^\sigma(A(\lambda+A)^{-1})^m x\| \le \frac{\mu^{m-\sigma}}{\lambda^{m-\sigma}}\|\mu^\sigma(A(\mu+A)^{-1})^{+1}mx\| + \\ + \frac{m}{((m-\sigma)p')^{\frac{1}{p'}}}\left(1-\frac{\mu^{m-\sigma}}{\lambda^{m-\sigma}}\right)\cdot \\ \cdot\left(\int_\mu^\lambda \|\tau^\sigma(A(\tau+A)^{-1})^{m+1}x\|^p\, d\tau/\tau\right)^{\frac{1}{p}}.$$

The second term tends to zero as $\mu \to \infty$ uniformly in $\lambda > \mu$ and so does the first term as $\lambda \to \infty$. Therefore, $x \in D_{\infty-}^\sigma$.
Since $\lambda^\sigma(A(\lambda+A)^{-1})^m x \in L^p(X) \cap L^{\infty-}(X)$, it is included each $L^q(X)$ with $q \in [p, \infty)$.
If $\tau < \sigma$, D_∞^σ is contained in D_q^τ for any q. Hence every D_q^σ is contained in D_q^τ.
Let $q \le \infty-$. Repeated application of the previous remark shows that $D_q^{\tau+m}$ is dense in D_q^τ for positive integer m. Since D_p^σ contains some $D_q^{\tau+m}$, it is dense in D_q^τ. □

In terms of spaces D_p^σ we can prove that the equality (1.5) holds for all $x \in D_1^\sigma$ for every $0 < \mathrm{Re} z \le \sigma$. Then the fractional power A^z for $\mathrm{Re} z > 0$ is the smallest closed extension of operators $A^z|_{D_1^\sigma}$ for all $\sigma \ge \mathrm{Re} z$.

Lemma 2.1.6 (Lemma 2.3 in [24]). *If m is an integer, $m > 0$, then*

$$A^m x = \lim_{N\to\infty} m\int_0^N \lambda^{m-1}(A(\lambda+A)^{-1})^{m+1}x\, d\lambda.$$

Proof. By (2.2) we have

$$m\int_0^N \lambda^{m-1}(A(\lambda+A)^{-1})^{m+1}x\, d\lambda = N^m(A(N+A)^{-1})^m x.$$

If $x \in D(A^m)$, $N^m(A(N+A)^{-1})^m x = (N(N+A)^{-1})^m A^m x$ tends to $A^m x$ as $N \to \infty$ by (1.6). Conversely, if $N^m(A(N+A)^{-1})^m x = A^m(N(N+A)^{-1})^m x$ converges to an element y, $x \in D(A^m)$ and $y = A^m x$. Then A^m is closed (see A. E. Taylor [38], Theorem 6.1) and $(N(N+A)^{-1})^m x$ converges to x. □

Proposition 2.1.7 (Proposition 2.4 in [24]). *If* $0 < \mathrm{Re}z < \sigma$, *there is a constant* $C(z,\sigma,p)$ *such that*

$$\|A^z x\| \le C(z,\sigma,p)\|\lambda^\sigma(A(\lambda+A)^{-1})^m x\|_{L^p(X)}^{\frac{\mathrm{Re}z}{\sigma}}\|x\|^{\frac{(\sigma-\mathrm{Re}z)}{\sigma}}$$

for all $x \in D_p^\sigma$.

Proof. Hölder inequality gives

$$\begin{aligned}\|A^z x\| \le & \left|\frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)}\right| \left[\int_0^t |\lambda^{z-1}\|(A(\lambda+A)^{-1})^m x\|\, d\lambda + \right.\\ & \left. + \int_t^\infty |\lambda^{z-\sigma}\|\lambda^\sigma(A(\lambda+A)^{-1})^m x\|\, d\lambda/\lambda\right] \le \\ & \le \left|\frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)}\right| \cdot \\ & \cdot \left[\frac{L^m t^{\mathrm{Re}z}}{\mathrm{Re}z}\|x\| + \frac{t^{\mathrm{Re}z-\sigma}}{((\sigma-\mathrm{Re}z)p')^{\frac{1}{p'}}}\|\lambda^\sigma(A(\lambda+A)^{-1})^n x\|_{L^p(X)}\right]\end{aligned}$$

Taking the minimum of the right-hand side over t varying in the set $(0,\infty)$, we obtain our statement. □

Proposition 2.1.8 (Proposition 2.5 in [24]). *If* $\mu > 0$ *then* $D_p^\sigma(A) = D_p^\sigma(\mu + A)$ *with equivalent norms.*

Proof. Let $x \in D_{p,m}^\sigma(A)$ with $m > \sigma$. Since

$$\|A^k(\lambda+\mu+A)^{-m}x\| \le C\|A^m(\lambda+\mu+A)^{-m}x\|^{\frac{k}{m}}\|(\lambda+\mu+A)^{-m}x\|^{\frac{m-k}{m}},$$

for $k \in \{1,2,\dots,m-1\}$, we have that

$$\lambda^\sigma((\mu+A)(\lambda+\mu+A)^{-1})^m x = \lambda^\sigma(\mu^m + m\mu^{m-1}A + \cdots + A^m)(\lambda+\mu+A)^{-m}x$$

belongs to $L^p(X)$. The converse is proved in the same way. □

Theorem 2.1.9 (Theorem 2.6 in [24]). *Let* $0 < \mathrm{Re}z < \sigma$. *Then* $x \in D_p^\sigma$ *if and only if* $x \in D(A^z)$ *and* $A^z x \in D_p^{\sigma-\mathrm{Re}z}$.

Proof. Let $x \in D_p^\sigma$ and $m > \sigma$. Clearly $x \in D(A^z)$. To estimate the integral

$$\lambda^{\sigma-\mathrm{Re}z}(A(\lambda+A)^{-1})^m A^z x =$$

$$= \frac{\Gamma(m)\lambda^{\sigma-\mathrm{Re}z}}{\Gamma(z)\Gamma(m-z)}\int_0^\infty \mu^{z-1}(A(\lambda+A)^{-1})^m(A(\mu+A)^{-1})^m x\, d\mu,$$

we split it into two parts. First

$$\|\lambda^{\sigma-\mathrm{Re}z}\int_0^\lambda \mu^{z-1}(A(\lambda+A)^{-1})^m(A(\mu+A)^{-1})^m x\,d\mu\| \leq$$
$$\leq \lambda^{\sigma-\mathrm{Re}z}\int_0^\lambda \mu^{\mathrm{Re}z-1}\,d\mu L^m\|(A(\lambda+A)^{-1})^m x\| =$$
$$= L^m(\mathrm{Re}z)^{-1}\lambda^\sigma\|(A(\lambda+A)^{-1})^m x\| \in L^p(X)$$

and the second part

$$\|\lambda^{\sigma-\mathrm{Re}z}\int_\lambda^\infty \mu^{z-1}(A(\lambda+A)^{-1})^m(A(\mu+A)^{-1})^m x\,d\mu\| \leq$$

$$\leq L^m\lambda^{\sigma-\mathrm{Re}z}\int_\lambda^\infty \mu^{\mathrm{Re}z-\sigma}\|\mu^\sigma(A(\mu+A)^{-1})^m x\|\,d\mu/\mu$$

also belongs to $L^p(X)$ because $\mathrm{Re}z-\sigma<0$.
Conversely, let $A^z x \in D_p^{\sigma-\mathrm{Re}z}$. If n is an integer greater than $\mathrm{Re}z$ we have

$$\|A^{n-\sigma}(\lambda+A)^{-n}\| \leq C\|A^n(\lambda+A)^{-n}\|^{\frac{n-\mathrm{Re}z}{n}}\|(\lambda+A)^{-n}\|^{\frac{\mathrm{Re}z}{n}} \leq C'\lambda^{-\mathrm{Re}z}.$$

Thus it follows that

$$\lambda^\sigma\|(A(\lambda+A)^{-1})^{m+n}x\| \leq \lambda^\sigma\|A^{n-z}(\lambda+A)^{-n}\|\|(A(\lambda+A)^{-1})^m A^z x\| \leq$$
$$\leq C'\lambda^{\sigma-\mathrm{Re}z}\|(A(\lambda+A)^{-1})^m A^z x\| \in L^p(X).$$

This completes the proof. □

Proposition 2.1.10 (Proposition 2.8 in [24]). *For every* $\mathrm{Re}z>0$

$$D_1^{\mathrm{Re}z} \subseteq D(A^z) \subseteq D_{\infty-}^{\mathrm{Re}z}.$$

Proof. It is enough to consider only the case $z=1$. The former inclusion is clear from Lemma 2.1.6. The latter follows from (1.6), because

$$\lambda(A(\lambda+A)^{-1})^2 x = \lambda(\lambda+A)^{-1}(1-\lambda(\lambda+A)^{-1})Ax \to 0$$

for $x \in D(A)$ as $\lambda\to\infty$. □

2.2 Definition of interpolation spaces $\mathbf{S}(\mathbf{p},\theta,\mathbf{X};\mathbf{p},\theta-\mathbf{1},\mathbf{Y})$

Let X and Y be Banach spaces contained in a Hausdorff vector space Z. Lions and Petree defined the mean space $S(p,\theta,X;p,\theta-1,Y)$, $p\in[1,\infty]$, $\theta\in(0,1)$, of X and Y as the space of the means

$$x=\int_0^\infty u(\lambda)\;d\lambda/\lambda,$$

where $u(\lambda)$ is a Z-valued function such that

$$\lambda^\theta u(\lambda)\in L^p(X)\quad\text{and}\quad\lambda^{\theta-1}u(\lambda)\in L^p(Y).\tag{2.3}$$

$S(p,\theta,X;p,\theta-1,Y)$ is a Banach space with the norm

$$\|x\|_{S(p,\theta,X;p,\theta-1,Y)}=$$

$$=\inf\left\{\max(\|\lambda^\theta u(\lambda)\|_{L^p(X)},\|\lambda^{\theta-1}u(\lambda)\|_{L^p(Y)})\colon x=\int_0^\infty u(\lambda)\;d\lambda/\lambda\right\}.$$

Theorem 2.2.1 (Theorem 3.1 in [24]). *$S(p,\theta,X;p,\theta-1,D(A^m))$, $p\in[1,\infty]$, $\theta\in(0,1)$, coincides with $D_p^{\theta m}(A)$.*

Proof. On the base of Proposition (2.1.8), we may assume, without loss of generality, that A has a bounded inverse. In particular, $D(A^m)$ is normed by $\|A^m x\|$. Further, if we change the variable by $\lambda'=\lambda^{\frac{1}{m}}$, condition (2.3) becomes

$$\lambda^{m\theta}u(\lambda)\in L^p(X)\quad\text{and}\quad\lambda^{m(\theta-1)}A^m u(\lambda)\in L^p(X).\tag{2.4}$$

Suppose $x\in D_p^\sigma$ and define

$$u(\lambda)=c\lambda^m A^m(\lambda+A)^{-2m}x,$$

where $c=\frac{\Gamma(2m)}{(\Gamma(m))^2}$. Then

$$\lambda^\sigma u(\lambda)=c(\lambda(\lambda+A)^{-1})^m\lambda^\sigma(A(\lambda+A)^{-1})^m x\in L^p(X)$$

and

$$\lambda^{\sigma-m}A^m u(\lambda)=c\lambda^\sigma(A(\lambda+A)^{-1})^{2m}x\in L^p(X).$$

Thus $u(\lambda)$ satisfies (2.4) with $\sigma = m\theta$. Moreover, it follows from Lemma (2.1.6) that

$$\int_0^\infty u(\lambda)\, d\lambda/\lambda = \frac{\Gamma(2m)}{(\Gamma(m))^2}\int_0^\infty \lambda^{m-1}(A(\lambda+A)^{-1})^{2m}A^{-m}x = x.$$

Therefore, x belongs to $S(p, \frac{\sigma}{m}, X; p, \frac{\sigma}{m} - 1, D(A^m))$.
Conversely, let $x \in S(p, \frac{\sigma}{m}, X; p, \frac{\sigma}{m} - 1, D(A^m))$ so that x is represented by integral $\int_0^\infty u(\lambda)\, d\lambda/\lambda$ with the integrand satisfying (2.4). Then

$$\lambda^\sigma(A(\lambda+A)^{-1})^m x = (A(\lambda+A)^{-1})^m\lambda^\sigma \int_\lambda^\infty \mu^{-\sigma}\mu^\sigma u(\lambda)\, d\lambda/\lambda +$$
$$+ (\lambda(\lambda+A)^{-1})^m\lambda^{\sigma-m}\int_0^\lambda \mu^{m-\sigma}\mu^{\sigma-m}A^m u(\lambda)\, d\lambda/\lambda.$$

Since both $(A(\lambda+A)^{-1})^m$ and $(\lambda(\lambda+A)^{-1})^m$ are uniformly bounded, $\lambda^\sigma(A(\lambda+A)^{-1})^m x$ belongs to $L^p(X)$, that is $x \in D_p^\sigma$. □

Now we define class of operators called *of type* $(\omega, M(\theta))$.

Definition 2.2.2. Let A be a closed operator densely defined in X. The operator A is said to be of *type* $(\omega, M(\theta))$ if there exists $\omega \in [0,\pi)$ and $M \geq 1$ such that $\Sigma_\omega = \{\lambda \in \mathbb{C}\colon |\arg\lambda| > \omega\} \subseteq \rho(A)$ and

$$\|\lambda(\lambda - A)^{-1}\| \leq M, \quad \text{for } \lambda < 0,$$

and, for all $\theta \in (\omega, \pi]$ there exists a number $M(\theta) \geq 1$ such that the inequality

$$\|\lambda(\lambda - A)^{-1}\| \leq M(\theta),$$

holds in $|\arg\lambda| \geq \theta$.

Remark 2.2.3. Any operator which is of the type $(\omega, M(\theta))$ with $0 \leq \omega < \pi$ has a dense domain and satisfies (1.1).

Theorem 2.2.4 (Theorem 3.2 in [24]). *Let A be an operator of type $(\omega, M(\theta))$. Then*

$$D_p^\sigma(A^z) = D_p^{\sigma z}(A), \quad 0 < z < \frac{\pi}{\omega}, \sigma > 0.$$

Proof. It is sufficient to prove it in case $0 < z < 1$, because otherwise we have $A = (A^z)^{\frac{1}{z}}$ with $0 < \frac{1}{z} < 1$. On the base of Theorem 2.1.9 we may also assume that σ is sufficiently small.
Basing on H. Komatsu [23] Proposition 10.2 we have

$$\lambda^\sigma A^z(\lambda+A^z)^{-1}x = \frac{\sin\pi z}{\pi}\int_0^\infty \frac{\lambda^{\sigma+1}\tau^{z-z\sigma}}{\lambda^2 + 2\lambda\tau^z\cos\pi z + \tau^{2z}}\tau^{z\sigma}A(\tau+A)^{-1}x\, d\tau/\tau.$$

Since the kernel

$$\frac{(\lambda^{-1}\tau^z)^{1-\sigma}}{1+2(\lambda^{-1}\tau^z)\cos\pi z+(\lambda^{-1}\tau^z)^2}, \quad 0<\sigma<1,$$

defines a bounded integral operator in $L^p(X)$, $D_p^{\sigma z}(A)$ is contained in $D_p^\sigma(A^z)$.

If $z=\frac{1}{m}$ with an odd integer m, we have conversely

$$D_p^\sigma(A^{\frac{1}{m}})\subseteq D_p^{\frac{\sigma}{m}}(A).$$

In fact, let $x\in D_p^\sigma(A^{\frac{1}{m}})$. Since

$$\lambda^\sigma A(\lambda^m+A)^{-1}x=\lambda^\sigma\prod_{i=1}^m(A^{\frac{1}{m}}(\varepsilon_i\lambda+A^{\frac{1}{m}})^{-1})x,$$

where ε_i are the roots of $(-\varepsilon)^m=-1$ with $\varepsilon_1=1$, and since

$$A^{\frac{1}{m}}(\varepsilon_i\lambda+A^{\frac{1}{m}})^{-1}, \quad i\in\{1,\dots,m\},$$

are uniformly bounded, then $\lambda^\sigma A(\lambda^m+A)^{-1}x\in L^p(X)$. Changing the variable by $\lambda'=\lambda^m$, we get $\lambda^{\frac{\sigma}{m}}A(\lambda+A)^{-1}x\in L^p(X)$.

In general case choose an odd number m such that $0<\frac{1}{m}<z$. Since $A^{\frac{1}{m}}=(A^z)^{\frac{1}{zm}}$, we have

$$D_p^{z\sigma}(A)\subseteq D_p^\sigma(A^z)\subseteq D_p^{z\sigma m}(A^{\frac{1}{m}})\subseteq D_p^{z\sigma}(A). \qquad \square$$

2.3 Complex interpolation space

We introduce the complex method of interpolation. It is considered by H. Triebel [40] and L. Tartar [37] among others.

Definition 2.3.1. For two Banach spaces X_0 and X_1, *the complex method of interpolation* consists in considering the space $F(X_0,X_1)$ of functions $f\colon\mathbb{C}\to X_0+X_1$, holomorphic on the open strip $0<\mathrm{Re}z<1$, continuous on the closed strip $0\le\mathrm{Re}z\le1$, and such that $f(iy)$ is bounded in X_0 and $f(1+iy)$ is bounded in X_1, equipped with the norm

$$\|f\|=\max\{\sup_{y\in\mathbb{R}}\|f(iy)\|_{X_0},\sup_{y\in\mathbb{R}}\|f(1+iy)\|_{X_1}\}.$$

For $0 < \theta < 1$, we define

$$[X_0, X_1]_\theta = \{a \in X_0 + X_1 \colon \exists_{f \in F(X_0,X_1)}\ a = f(\theta)\},$$

with the norm

$$\|a\|_{[X_0,X_1]_\theta} = \inf_{f(\theta)=a} \|f\|.$$

Of course such a space contains $X_0 \cap X_1$, as we can take f to be a constant function taking its value in $X_0 \cap X_1$. So we have inclusions

$$X_0 \cap X_1 \subseteq [X_0, X_1]_\theta \subseteq X_0 + X_1. \tag{2.5}$$

Now we prove one very important property of an interpolation space, called *the interpolation property.* It says that $[\cdot,\cdot]_\theta$ is the *interpolation functor.*

Lemma 2.3.2. *Let X_0, X_1, Y_0, Y_1 be Banach spaces and $0 < \theta < 1$. There exists a constant $C > 0$ such that for all $A \in \mathcal{L}(X_0, Y_0) \cap \mathcal{L}(X_1, Y_1)$,*

$$\|A\|_{\mathcal{L}([X_0,X_1]_\theta,[Y_0,Y_1]_\theta)} \le C\|A\|_{\mathcal{L}(X_0,Y_0)}\|A\|_{\mathcal{L}(X_1,Y_1)}.$$

Proof. If $A \in \mathcal{L}(X_0, Y_0) \cap \mathcal{L}(X_1, Y_1)$, then $g(z) = Af(z)$ satisfies a property similar to f with the spaces Y_0 and Y_1, so that we has

$$\|Aa\|_{[Y_0,Y_1]_\theta} \le \max\{\|A\|_{\mathcal{L}(X_0,Y_0)}, \|A\|_{\mathcal{L}(X_1,Y_1)}\}\|a\|_{[X_0,X_1]_\theta}.$$

We may replace $\max\{\|A\|_{\mathcal{L}(X_0,Y_0)}, \|A\|_{\mathcal{L}(X_1,Y_1)}\}$ by $\|A\|^{1-\theta}_{\mathcal{L}(X_0,Y_0)} \cdot \|A\|^{\theta}_{\mathcal{L}(X_1,Y_1)}$, by considering $g(z) = e^{-s\theta+sz}Af(z)$ instead, which results in getting the quantity $\max\{e^{-s\theta}\|A\|_{\mathcal{L}(X_0,Y_0)}, e^{s(1-\theta)}\|A\|_{\mathcal{L}(X_1,Y_1)}\}$ appear, and then one minimizes in s by taking $e^s = \frac{\|A\|_{\mathcal{L}(X_0,Y_0)}}{\|A\|_{\mathcal{L}(X_1,Y_1)}}$. □

Next we introduce some other properties of this interpolation functor.

Theorem 2.3.3. *Let X_0, X_1 be Banach spaces and $0 < \theta < 1$.*

(a) $[X_0, X_1]_\theta = [X_1, X_0]_{1-\theta}$.
(b) *If $X_0 \subseteq X_1$ and $0 < \theta < \theta_1 < 1$, then*

$$X_0 \subseteq [X_0, X_1]_\theta \subseteq [X_0, X_1]_{\theta_1} \subseteq X_1.$$

(c) *If $X_0 = X_1$, then $[X_0, X_1]_\theta = X_0 = X_1$.*
(d) *There exists a constant $C_\theta > 0$ such that for all $a \in X_0 \cap X_1$ we have*

$$\|x\|_{[X_0,X_1]_\theta} \le C_\theta \|x\|^{1-\theta}_{X_0} \|x\|^{\theta}_{X_1}.$$

Proof.

(a) Let $f \in F(X_0, X_1)$. Then the function $g(z) = f(1-z)$ belongs to $F(X_1, X_0)$ and $\inf_{f(\theta)=a} \|f\| = \inf_{f(1-\theta)=a} \|g\|$.
(b) Let $X_0 \subseteq X_1$. From (2.5) we have

$$X_0 \subseteq [X_0, X_1]_\theta \subseteq X_1.$$

Because of $0 < \theta < \theta_1 < 1$, there exists a $0 < \lambda < 1$ such that $\theta = \lambda\theta_1$. Till the end of the proof we have to show that

$$[X_0, X_1]_\theta \subseteq [X_0, [X_0, X_1]_{\theta_1}]_\lambda.$$

Let $a \in [X_0, X_1]_\theta$ and $f \in F(X_0, X_1)$ be such that $f(\theta) = a$. Define a new function $g(z) = f(\theta_1 z)$. Of course $g(\lambda) = f(\theta_1\lambda) = a$. Function g is holomorphic in the strip $0 < \mathrm{Re} z < 1$ and continuous in its closure. $g(iy) = f(\theta_1 iy)$ is bounded in X_0 and because of $\|f(\theta_1+\theta_1 iy)\|_{[X_0,X_1]_{\theta_1}} \leq \|f\|$, function $g(1+iy)$ is bounded in $[X_0, X_1]_{\theta_1}$. Thus $g \in F(X_0, [X_0, X_1]_{\theta_1})$ and $\|g\| \leq \|f\|$.
(c) Follows directly from (b). □

Corollary 2.3.4. *For* $0 \leq \theta_0 < \theta_1 \leq 1$ *and* $0 < \lambda < 1$ *the following inclusion holds*

$$[X_0, X_1]_{\theta_0(1-\lambda)+\theta_1\lambda} \subseteq [[X_0, X_1]_{\theta_0}, [X_0, X_1]_{\theta_1}]_\lambda.$$

Proof. The proof is the same as for the point (b) of the previous Theorem if we take $g(z) = f(\theta_0(1-z) + \theta_1 z)$. □

2.4 Another definition of interpolation spaces; Real interpolation space

In this section we introduce methods of the real interpolation. We present facts without proofs only. We refer to the papers of A. Lunardi [29], H. Triebel [40] or L. Tartar [37] for more details.

2.4.1 The K-method

Let X and Y be Banach spaces with $Y \subseteq X$, so that there exists $c > 0$ such that

$$\|y\|_X \le c\|y\|_Y, \quad \text{for all } y \in Y.$$

We describe briefly the construction of a family of intermediate spaces between X and Y, called *real interpolation spaces*, and denoted by $(X,Y)_{\theta,p}$, $(X,Y)_\theta$, with $0 < \theta \le 1$, $1 \le p \le \infty$. We follow the so called *K-method*. Throughout this section we set $\frac{1}{\infty} = 0$.

Definition 2.4.1 (Definition 1.2.1 in [29]). For every $x \in X$ and $t > 0$, set

$$K(t,x) = K(t,x,X,Y) = \inf_{\substack{x=a+b,\\ a\in X, b\in Y}} (\|a\|_X + t\|b\|_Y).$$

From Definition 2.4.1 it follows immediately that for every $t > 0$ and $x \in X$ we have

$$\begin{aligned} &\text{(i)} \;\; \min\{1,t\}K(1,x) \le K(t,x) \le \max\{1,t\}K(1,x), \\ &\text{(ii)} \;\; K(t,x) \le \|x\|_X. \end{aligned} \tag{2.6}$$

Definition 2.4.2 (Definition 1.2.2 in [29]). Let $\theta \in (0,1]$, $p \in [1,\infty]$, and set

$$\begin{aligned} (X,Y)_{\theta,p} &= \{x \in X \colon u(t) = t^{-\theta-\frac{1}{p}}K(t,x) \in L^p(0,\infty)\}, \\ (X,Y)_\theta &= \{x \in X \colon \lim_{t\to 0} t^{-\theta}K(t,x) = 0\}, \end{aligned}$$

and

$$\|x\|_{\theta,p} = \|t^{-\theta-\frac{1}{p}}K(t,x)\|_{L^p(0,\infty)}$$

It is easily seen that the mapping $x \mapsto \|x\|_{\theta,p}$ is a norm in $(X,Y)_{\theta,p}$. Since $t \mapsto K(t,x)$ is bounded, it is clear that only the behaviour near $t = 0$ of $t^{-\theta}K(t,x)$ is significant in the definition of $(X,Y)_{\theta,p}$ and $(X,Y)_\theta$. Indeed, one could replace the half line $(0,\infty)$ by any interval $(0,a)$ in Definition 2.4.2, obtaining equivalent norms.

For $\theta = 1$, from the first inequality in (2.6) (i), we get

$$(X,Y)_1 = (X,Y)_{1,p} = \{0\}, \quad p < \infty.$$

Therefore, from now on we shall consider the cases $(\theta,p) \in (0,1) \times [1,\infty]$ and $(\theta,p) = (1,\infty)$.

If $X = Y$, then $K(t,x) = \min\{1,t\}\|x\|$. Therefore, as one can expect, $(X,X)_{\theta,p} = (X,X)_{1,\infty} = X$ for $\theta \in (0,1)$, $p \in [1,\infty]$, and

$$\|x\|_{(X,X)_{\theta,p}} = \left(\frac{1}{p\theta(1-\theta)}\right)^{\frac{1}{p}} \|x\|_X, \quad \theta \in (0,1), p \in [1,\infty),$$
$$\|x\|_{(X,X)_{\theta,\infty}} = \|x\|_X, \quad \theta \in (0,1].$$

Some inclusion properties are stated below.

Proposition 2.4.3 (Proposition 1.2.3 in [29]). *For $\theta \in (0,1)$, $1 \le p_1 \le$ $\le p_2 \le \infty$, we have*

$$Y \subseteq (X,Y)_{\theta,p_1} \subseteq (X,Y)_{\theta,p_2} \subseteq (X,Y)_\theta \subseteq (X,Y)_{\theta,\infty} \subseteq \overline{Y}.$$

For $0 < \theta_1 < \theta_2 \le 1$ we have

$$(X,Y)_{\theta_2,\infty} \subseteq (X,Y)_{\theta_1,1}.$$

Proposition 2.4.4 (Proposition 1.2.4 in [29]). *$((X,Y)_{\theta,p}, \|\cdot\|_{\theta,p})$ is a Banach space.*

Corollary 2.4.5 (Corollary 1.2.5 in [29]). *For $\theta \in (0,1]$, $(X,Y)_\theta$ is a Banach space endowed with the norm of $(X,Y)_{\theta,\infty}$.*

Proposition 2.4.6 (Corollary 1.2.7 in [29]). *For $\theta \in (0,1)$, $p \in [1,\infty]$ and for $(\theta,p) = (1,\infty)$ there is a constant $c > 0$ such that*

$$\|y\|_{\theta,p} \le c\|y\|_X^{1-\theta}\|y\|_Y^\theta, \quad y \in Y.$$

Now we present the definition of the space J_θ

Definition 2.4.7 (Definition 1.2.2 in [29]). Let $\theta \in [0,1]$. A Banach space E such that $Y \subseteq E \subseteq X$ is said to belong to the class between X and Y if there is a constant c such that

$$\|x\|_E \le c\|x\|_X^{1-\theta}\|x\|_Y^\theta, \quad x \in Y.$$

In this case we write $E \in J_\theta(X,Y)$.

Below we inmtroduce some important examples.

Example 2.4.8 (Proposition 1.1.2 in [29]). Let k, m be positive integers such that $k < m$. Then:

(i) $C_b^k(\mathbb{R},X)$ belongs to the class $J_{\frac{k}{m}}$ between $C_b(\mathbb{R},X)$ and $C_b^m(\mathbb{R},X)$;

(ii) $C^k(\mathbb{R}^n)$ belongs to the class $J_{\frac{k}{m}}$ between $C(\mathbb{R}^n)$ and $C^m(\mathbb{R}^n)$;

(iii) if Ω is an open set in $\mathbb{R}^n$ with uniformly C^m boundary, then $C^k(\overline{\Omega})$ belongs to the class $J_{\frac{k}{m}}$ between $C(\overline{\Omega})$ and $C^m(\overline{\Omega})$.

The statement of Proposition 2.4.6 can be rephrased by saying that every $(X,Y)_{\theta,p}$belongs to $J_\theta(X,Y)$. In particular, $(X,Y)_{1,p}$ belongs to $J_\theta(X,Y)$. Later we will see that in fact space E belongs to the class $J_\theta(X,Y)$ if and only if $(X,Y)_{\theta,1}$ is continuously embedded in E.

2.4.2 The trace method

We describe now another construction of real interpolation spaces, which is one of the most common in the literature and which will be useful for proving the other properties.

Definition 2.4.9 (Definition 1.2.8 in [29]). For $\theta \in [0,1)$, $p \in [1,\infty]$, set

$$V(p,\theta,Y,X) =$$

$$= \{u\colon \mathbb{R}_+ \to X | u_\theta = t^{\theta-\frac{1}{p}}u(t) \in L^p(0,\infty;Y), v_\theta = t^{\theta-\frac{1}{p}}u'(t) \in L^p(0,\infty;X)\},$$

with

$$\|u\|_{V(p,\theta,Y,X)} = \|u_\theta\|_{L^p(0,\infty;Y)} + \|v_\theta\|_{L^p(0,\infty;X)}.$$

Moreover, for $p = \infty$ we define a subspace of $V(\infty,\theta,Y,X)$ by

$$V_0(\infty,\theta,Y,X) = \{u \in V(\infty,\theta,Y,X)\colon \lim_{t\to 0}\|t^\theta u(t)\|_X = \lim_{t\to 0}\|t^\theta u'(t)\|_Y = 0\}.$$

It is not difficult to see that $V(p,\theta,Y,X)$ is a Banach space endowed with the norm $\|\cdot\|_{V(p,\theta,Y,X)}$, and that $V_0(\infty,\theta,Y,X)$ is a closed subspace of $V(\infty,\theta,Y,X)$. Moreover, if $\theta < 1$, any function belonging to $V(p,\theta,Y,X)$ has a X-valued continuous extension at $t = 0$. Indeed, for $0 < s < t$ from the equality $u(t) - u(s) = \int_s^t u'(\sigma)\, d\sigma$, for $p \in (1,\infty)$, it follows

$$\begin{aligned}\|u(t) - u(s)\|_X &\le \left(\int_s^t \|\sigma^\theta u'(\sigma)\|_X^p \, d\sigma/\sigma\right)^{\frac{1}{p}} \left(\int_s^t \sigma^{-(\frac{\theta-1}{p})q}\, d\sigma\right)^{\frac{1}{q}}\\ &\le \|u\|_{V(p,\theta,Y,X)}[q(1-\theta)]^{-\frac{1}{q}}(t^{q(1-\theta)} - s^{q(1-\theta)})^{\frac{1}{q}},\end{aligned}$$

with $q = \frac{p}{1-p}$. By using similar argumentation, we see that if $p = 1$ or $p = \infty$, then u is Lipschitz continuous (respectively, $(1-\theta)$-Hölder continuous) near $t = 0$.

In this section we shall use the Hardy-Young inequalities, which hold for every positive measurable function ϕ: $(0,a) \to \mathbb{R}$, $a \in (0,\infty]$, and every $\alpha > 0$, $p \geq 1$:

$$\begin{aligned} &\text{(i)} \int_0^a t^{-\alpha p} \left(\int_0^t \phi(s)\, ds/s \right)^p dt/t \leq \frac{1}{\alpha^p} \int_0^a s^{-\alpha p} \phi(s)^p \, ds/s \\ &\text{(ii)} \int_0^a t^{\alpha p} \left(\int_t^a \phi(s)\, ds/s \right)^p dt/t \leq \frac{1}{\alpha^p} \int_0^a s^{\alpha p} \phi(s)^p \, ds/s \end{aligned} \tag{2.7}$$

We shall use the following consequence of inequality (2.7) (i).

Corollary 2.4.10 (Corollary 1.2.9 in [29]). *Let u be a function such that $t \mapsto u_\theta(t) = t^{\theta - \frac{1}{p}} u(t)$ belongs to $L^p(0,a;X)$, with $a \in (0,\infty]$, $\theta \in (0,1)$ and $p \in [1,\infty]$. Then also the mean value*

$$v(t) = \frac{1}{t} \int_0^t u(s)\, ds, \quad t > 0$$

has the same property, and setting $v_\theta(t) = t^{\theta - \frac{1}{p}} v(t)$ we obtain

$$\|v_\theta\|_{L^p(0,a;X)} \leq \frac{1}{1-\theta} \|u_\theta\|_{L^p(0,a;X)}.$$

With the aid of Corollary 2.4.10 we are able to characterize the real interpolation spaces as trace spaces.

Proposition 2.4.11 (Proposition 1.2.10 in [29]).
For $(\theta,p) \in (0,1) \times [1,\infty] \cup \{(1,\infty)\}$, $(X,Y)_{\theta,p}$ is the set of the traces at $t - 0$ of the functions in $V(p, 1-\theta, Y, X)$, and the norm

$$\|x\|^T_{\theta,p} = \inf\{\|u\|_{V(p,1-\theta,Y,X)} \colon x = u(0), u \in V(p,1-\theta,Y,X)\}$$

is an equivalent norm of $(X,Y)_{\theta,p}$. Moreover, for $\theta \in (0,1)$, $(x,Y)_\theta$ is the set of the traces at $t = 0$ of the functions in $V_0(\infty, 1-\theta, Y, X)$.

By Proposition 2.4.11, if $x \in (X,Y)_{\theta,p}$ or $x \in (X,Y)_\theta$, then x is the trace at $t = 0$ of a function u belonging to $L^p(a,b;Y) \cap W^{1,p}(a,b;X)$ for $0 < a < b$. A. Lunardi [29] states in Remark 1.2.11 that it is possible to find a more regular function $v \in V(p,1-\theta,Y,X)$ (or $v \in V_0(p,1-\theta,Y,X)$), such that $v(0) = x$, which belongs to $W^{1,p}(a,b;Y) \cap W^{2,p}(a,b;X)$.

By means of the trace method it is easy to prove some important density properties.

Proposition 2.4.12 (Proposition 1.2.12 in [29]). *Let $\theta \in (0,1)$. For $p \in [1,\infty)$, Y is dense in $(X,Y)_{\theta,p}$. For $p = \infty$, $(X,Y)_\theta$ is the closure of Y in $(X,Y)_{\theta,p}$.*

In the previous section we have seen that every $(X,Y)_{\theta,p}$ belongs to $J_\theta(X,Y)$. In particular, $(X,Y)_{\theta,1}$ belongs to $J_\theta(X,Y)$. Now we can characterize all the spaces in the class $J_\theta(X,Y)$.

Proposition 2.4.13 (Proposition 1.2.13 in [29]). *Let $\theta \in (0,1)$, and let E be a Banach space such that $Y \subseteq E \subseteq X$. The following statements are equivalent:*

(i) *E belongs to the class $J_\theta(X,Y)$,*
(ii) *$(X,Y)_{\theta,1} \subseteq E$.*

2.4.3 The Reiteration Theorem

We need some preliminaries about certain classes of intermediate spaces between X and Y. We have introduced the class J_θ in the previous section, and we have shown that a Banach space E such that $Y \subseteq E \subseteq X$ belongs to $J_\theta(X,Y)$ if and only if $(X,Y)_{\theta,1}$ is continuously embedded in E. Now we define another class of intermediate spaces.

Definition 2.4.14 (Definition 1.2.14 in [29]). Let E be a Banach space such that $Y \subseteq E \subseteq X$, and let $\theta \in [0,1]$. E is said to belong to the class K_θ between X and Y if there exists $k > 0$ such that

$$K(t,x) \le kt^\theta \|x\|_E, \quad x \in E, t > 0$$

In other words, E belongs to the class K_θ if and only if it is continuously embedded in $(X,Y)_{\theta,\infty}$. In that case, we write $E \in K_\theta(X,Y)$.

By Definition 2.4.14 and Proposition 2.4.13, a space E belongs to $K_\theta(X,Y) \cap J_\theta(X,Y)$ if and only if

$$(X,Y)_{\theta,1} \subseteq E \subseteq (X,Y)_{\theta,\infty}.$$

Now we are able to state the Reiteration Theorem.

Theorem 2.4.15 (Theorem 1.2.15 in [29]). *Let $0 \le \theta_0 \le \theta_1 \le 1$. Fix $\theta \in (0,1)$ and set $\omega = (1-\theta)\theta_0 + \theta\theta_1$. The following statemants hold true.*

(i) *If E_i belongs to the class K_{θ_i} ($i \in \{0,1\}$) between X and Y, then*

$$(E_0,E_1)_{\theta,p} \subseteq (X,Y)_{\omega,p}, \quad \textit{for } p \in [1,\infty], \quad (E_0,E_1)_\theta \subseteq (X,Y)_\omega.$$

(ii) *If E_i belongs to the class J_{θ_i} ($i \in \{0,1\}$) between X and Y, then*

$$(X,Y)_{\omega,p} \subseteq (E_0,E_1)_{\theta,p}, \quad \textit{for } p \in [1,\infty], \quad (X,Y)_{\omega}(E_0,E_1)_{\theta}.$$

Consequently, if E_i belongs to $K_{\theta_i}(X,Y) \cap J_{\theta_i}(X,Y)$, then

$$(E_0,E_1)_{\theta,p} = (X,Y)_{\omega,p}, \quad \textit{for } p \in [1,\infty], \quad (E_0,E_1)_{\theta} = (X,Y)_{\omega},$$

with equivalence of the corresponding norms.

Remark 2.4.16 (Remark 1.2.16 in [29]). By proposition 2.4.3, $(X,Y)_{\theta,p}$ and $(X,Y)_{\theta}$ belong to $K_\theta(X,Y) \cap J_\theta(X,Y)$ for $\theta \in (0,1)$ and $p \in [1,\infty]$. The Reiteration Theorem yields

$$\begin{aligned}
((X,Y)_{\theta_0,q_0},(X,Y)_{\theta_1,q_1})_{\theta,p} &= (X,Y)_{(1-\theta)\theta_0+\theta\theta_1,p},\\
((X,Y)_{\theta_0},(X,Y)_{\theta_1,q})_{\theta,p} &= (X,Y)_{(1-\theta)\theta_0+\theta\theta_1,p},\\
((X,Y)_{\theta_0,q},(X,Y)_{\theta_1})_{\theta,p} &= (X,Y)_{(1-\theta)\theta_0+\theta\theta_1,p},
\end{aligned}$$

for $\theta_0,\theta_1 \in (0,1)$, $p,q,q_0,q_1 \in [1,\infty]$. Moreover, since X belongs to $K_0(X,Y) \cap J_0(X,Y)$ and Y belongs to $K_1(X,Y) \cap J_1(X,Y)$ between X and Y, then

$$((X,Y)_{\theta_0,q},Y)_{\theta,p} = (X,Y)_{(1-\theta)\theta_0+\theta\theta_1}, \quad ((X,Y)_{\theta_0},Y)_{\theta} = (X,Y)_{(1-\theta)\theta_0+\theta},$$

and

$$(X,(X,Y)_{\theta_1,q})_{\theta,p} = (X,Y)_{\theta_1\theta,p} \quad (X,(X,Y)_{\theta_1})_{\theta_1\theta},$$

for $\theta,\theta_0,\theta_1 \in (0,1)$, $p,q \in [1,\infty]$.

2.4.4 Some examples

We will provide here the examples of some important interpolation spaces.

Theorem 2.4.17 (Theorem 1.2.17 in [29]). *For $\theta \in (0,1)$, $m \in \mathbb{N}$, it holds*

$$(C(\mathbb{R}^n),C^m(\mathbb{R}^n))_{\theta,\infty} = C^{\theta m}(\mathbb{R}^n),$$

with equivalence of the respective norms.

Corollary 2.4.18 (Corollary 1.2.18 in [29]). *For $0 \le \theta_1 < \theta_2$, $\sigma \in (0,1)$ it holds*

$$(C^{\theta_1}(\mathbb{R}^n),C^{\theta_2}(\mathbb{R}^n))_{\sigma,\infty} = C^{\theta_1+\sigma(\theta_2-\theta_1)}(\mathbb{R}^n).$$

Theorem 2.4.17 and Corollary 2.4.18 yields the characterization of other interpolation spaces between spaces of functions defined in arbitrary smooth domains.

Corollary 2.4.19 (Corollary 1.2.19 in [29]). *Let* $0 \leq \theta_1 < \theta_2$ *and* $\sigma \in (0,1)$. *If* Ω *is an open set in* $\mathbb{R}^n$ *with uniformly* C^{θ_2} *boundary, then*

$$(C^{\theta_1}(\overline{\Omega}), C^{\theta_2}(\overline{\Omega}))_{\sigma,\infty} = C^{\theta_1+\sigma(\theta_2-\theta_1)}(\overline{\Omega}),$$

with equivalence of the respective norms, and if $\theta_1 + \sigma(\theta_2 - \theta_1)$ *is not an integer, then*

$$(C^{\theta_1}(\overline{\Omega}), C^{\theta_2}(\overline{\Omega}))_{\sigma} = h^{\theta_1+\sigma(\theta_2-\theta_1)}(\overline{\Omega}).$$

Chapter 3

Infinitesimal generators of semi-groups

In this chapter we consider infinitesimal generators of bounded semi-groups and analytic bounded semi-groups. The results are taken from the papers of H. Komatsu [23] and [24].

3.1 Infinitesimal generators of bounded semi-groups

Throughout this section we assume that $(X, \|\cdot\|)$ is a Banach space and $\{T_t\}_{t\geq 0}$ is a bounded strongly continuous semi-group of operators in X and $-A$ is its infinitesimal generator:

$$T_t = \exp(-tA), \quad \|T_t\| \leq M.$$

A is an operator of type $(\frac{\pi}{2}, M(\theta))$.

Definition 3.1.1. Let $0 < \sigma < m$ where $\sigma \in \mathbb{R}$ and $m \in \mathbb{N}$ and let $p \in [1, \infty]$. We denote by $C^\sigma_{p,m} = C^\sigma_{p,m}(A)$ the set of all elements $x \in X$ such that

$$t^{-\sigma}(I - T_t)^m x \in L^p(X). \tag{3.1}$$

As is easily seen, $C^\sigma_{p,m}$ is a Banach space with the norm

$$\|x\|_{C^\sigma_{p,m}} = \|x\| + \|t^{-\sigma}(I - T_t)^m x\|_{L^p(X)}.$$

Since $(I - T_t)^m$ is uniformly bounded, condition (3.1) is equivalent to the fact that $t^{-\sigma}(I - T_t)^m x$ belongs to $L^p(X)$ near to origin.

Proposition 3.1.2 (Proposition 4.2 in [24]). *If $x \in C^\sigma_{p,m}$, then x belongs to $D(A^z)$ for all $0 < \mathrm{Re} z < \sigma$ and*

$$A^z x = \frac{1}{K_{z,m}} \int_0^\infty t^{-z-1}(I - T_t)^m x \, dt, \quad 0 < \mathrm{Re} z < \sigma$$

where

$$K_{z,m} = \int_0^\infty t^{-z-1}(1-e^{-t})^m \, dt.$$

Theorem 3.1.3 (Theorem 4.3 in [24]). *$C^\sigma_{p,m}$ coincides with D^σ_p with equivalent norms.*

Theorem 3.1.4 (Theorem 4.4 in [24]). *Let $0 < \mathrm{Re} z < m$. If there is a sequence $\varepsilon_j \to 0$ such that the limit*

$$y = \lim_{j\to\infty} \frac{1}{K_{z,m}} \int_{\varepsilon_j}^\infty t^{-z-1}(I-T_t)^m x \, dt$$

exists in a weak topology in X, i.e. the topology generated by all linear continuous functionals on X, then $x \in D(A^z)$ and $y = A^z x$.
Conversely, if $x \in D(A^z)$, then

$$A^z x = \lim_{\varepsilon\to 0} \frac{1}{K_{z,m}} \int_{\varepsilon}^\infty t^{-z-1}(I-T_t)^m x \, dt$$

exists in strong topology in X.

3.2 Infinitesimal generators of bounded analytic semi-groups

Let $(T_t)_{t\geq 0}$ be a semi-group of operators analytic in a sector $|\arg t| < \frac{\pi}{2} - \omega$, $\omega \in [0, \frac{\pi}{2})$, uniformly bounded in each smaller sector $|\arg t| \leq \frac{\pi}{2} - \omega - \varepsilon$, $\varepsilon > 0$. We call such a semi-group *a bounded analytic semi-group.*

It is known that the negative of an operator A generates a bounded analytic semi-group if and only if A is of the type $(\omega, M(\theta))$ for some $\omega \in [0, \frac{\pi}{2})$. A bounded strongly continuous semi-group T_t has a bounded analytic extension if there is a complex number z with $\mathrm{Re} z > 0$ such that

$$\|A^z T_t\| \leq C t^{-\mathrm{Re} z}, \quad t > 0 \tag{3.2}$$

with a constant C independent of t. Conversely, if T_t is a bounded analytic, (3.2) holds for all $\mathrm{Re} z > 0$.

Definition 3.2.1. Let $0 < \sigma < \mathrm{Re} z_1$ and $p \in [1, \infty]$. We denote by $B^\sigma_{p,z_1} = B^\sigma_{p,z_1}(A)$ the set of all $x \in X$ such that

$$t^{\mathrm{Re} z_1 - \sigma} A^{z_1} T_t x \in L^p(X).$$

B^σ_{p,z_1} is a Banach space with the norm

$$\|x\|_{B^\sigma_{p,z_1}} = \|x\| + \|t^{\mathrm{Re} z_1 - \sigma} A^{z_1} T_t x\|_{L^p(X)}.$$

Proposition 3.2.2 (Proposition 5.2 in [24]). *Let* $0 < \mathrm{Re} z < \sigma$. *Then every* $x \in B^\sigma_{p,z_1}$ *belongs to* $D(A^z)$ *and*

$$A^z x = \frac{1}{\Gamma(z_1 - z)} \int_0^\infty t^{z_1 - z - 1} A^{z_1} T_t x \, dt,$$

where the integral converges absolutely.

Theorem 3.2.3 (Theorem 5.3 in [24]). B^σ_{p,z_1} *coincides with* D^σ_p. *In particular,* B^σ_{p,z_1} *is independent of* z_1.

Theorem 3.2.4 (Theorem 5.4 in [24]). *Let* $0 < \mathrm{Re} z < \mathrm{Re} z_1$. *If*

$$y = \lim_{\varepsilon_j \to 0} \frac{1}{\Gamma(z_1 - z)} \int_{\varepsilon_j}^\infty t^{z_1 - z - 1} A^{z_1} T_t x \, dt$$

exists in a weak topology, i.e. the topology generated by all linear continuous functionals on X, *then* $x \in D(A^z)$ *and* $y = A^z x$. *If* $x \in D(A^z)$, *then*

$$A^z x = \lim_{\varepsilon \to 0} \frac{1}{\Gamma(z_1 - z)} \int_{\varepsilon}^\infty t^{z_1 - z - 1} A^{z_1} T_t x \, dt$$

exists in strong topology in X.

Chapter 4

Scales of Banach Spaces

To start with a simple example let us consider first a Hilbert space H with a countable base supplied by a selfadjoint operator.

Let H be a real separable Hilbert space and $A\colon D(A) \subseteq H \to H$ be an unbounded, closed, positive selfadjoint, linear operator with compact resolvent. Let $\sigma(A) = \{\lambda_n\}_{n\in\mathbb{N}}$ be the non-decreasing sequence of the eigenvalues of A counted with their multiplicity, converging to $+\infty$, with $\lambda_1 > 0$, and let $\{e_n\}_{n\in\mathbb{N}}$ be a Hilbert basis of eigenvectors of A; so that $Ae_n = \lambda_n e_n$. Thus the spectral representation of A is given by: if $u = \sum_{n=1}^{\infty} u_n e_n \in D(A) \subseteq H$ then $Au = \sum_{n=1}^{\infty} \lambda_n u_n e_n \in H$ and

$$D(A) = \left\{ u \in H\colon u = \sum_{n=1}^{\infty} u_n e_n \text{ and } \sum_{n=1}^{\infty} |\lambda_n|^2 |u_n|^2 < \infty \right\};$$

for more details see K. Yosida [41].

A. Rodriguez Bernal in his paper [32] constructs the fractional powers of such operator A (see chapter 1) and gives their following characterization.

For $\alpha \in [0, \infty)$ we denote

$$X^\alpha = \left\{ u \in H\colon u = \sum_{n=1}^{\infty} u_n e_n \text{ and } \sum_{n=1}^{\infty} |\lambda_n|^{2\alpha} |u_n|^2 < \infty \right\}$$

and $A^\alpha\colon D(A^\alpha) = X^\alpha \to X^0 = H$ is defined by

$$A^\alpha u = \sum_{n=1}^{\infty} \lambda_n^\alpha u_n e_n, \quad \text{if} \quad u = \sum_{n=1}^{\infty} u_n e_n \in D(A^\alpha).$$

The expression

$$\|u\|_\alpha = \|A^\alpha u\|_H = \left(\sum_{n=1}^{\infty} |\lambda_n|^{2\alpha} |u_n|^2 \right)^{\frac{1}{2}}$$

defines a hilbertian norm in X^α.

Domains of the operators A^α, $\alpha > 0$ can be considered in more general framework. Let $\alpha \in \mathbb{R}$. We define $\mathbb{X}^\alpha$ as the linear space of real sequences $x = (x_n)_{n\in\mathbb{N}}$ such that $\sum\limits_{n=1}^{\infty} |\lambda_n|^{2\alpha}|x_n|^2 < \infty$, endowed with the norm $|\!|\!|x|\!|\!|_\alpha = \left(\sum\limits_{n=1}^{\infty} |\lambda_n|^{2\alpha}|x_n|^2\right)^{\frac{1}{2}}$. We can observe that when $\alpha > 0$, then there exists an isometry between the spaces $(X^\alpha, \|\cdot\|_\alpha)$ and $(\mathbb{X}^\alpha, |\!|\!|\cdot|\!|\!|_\alpha)$. To simplify, both X^α and $\mathbb{X}^\alpha$ will be denoted as X^α. Now our aim is to present basic properties of the spaces X^α.

Proposition 4.0.1.

(i) *$(X^\alpha, \|\cdot\|_\alpha)$ is a Hilbert space for every $\alpha \in \mathbb{R}$.*
(ii) *If $\alpha \geq \beta$, then $X^\alpha \subseteq X^\beta$ with continuous (compact if $\alpha > \beta$) and dense injection.*
(iii) *If $\alpha, \beta \in \mathbb{R}$ and $\theta \in [0,1]$, then for every $x \in X^\gamma$ where $\gamma = \max(\alpha,\beta)$, we have*

$$\|x\|_{\theta\alpha+(1-\theta)\beta} \leq \|x\|_\alpha^\theta \|x\|_\beta^{1-\theta}.$$

The proof of this Proposition can be found in A. Rodriguez Bernal [32], p. 6.

A sequence of the space satisfying properties (i)–(iii) is called *a scale of Hilbert spaces.* In this chapter we will construct *a scale of Banach spaces* which will have the same properties.

4.1 Inductive Limits and Projective Limits of Sequences of Banach Spaces

Let $\{(E_n, \|\cdot\|_n) \colon n \in \mathbb{N}_0\}$ be a sequence of Banach spaces such that $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}_0$ and the embeddings are all continuous, i.e. for each $n \in \mathbb{N}_0$ there exists a constant $M_n > 0$ such that

$$\|u\|_{n+1} \leq M_n \|u\|_n, \quad \text{for all } u \in E_n. \tag{4.1}$$

In this case there always exists a corresponding sequence of new norms $|\cdot|_n$ on E_n which are equivalent to $\|\cdot\|_n$ respectively and which are monotonously decreasing, i.e.

$$|u|_{n+1} \leq |u|_n, \quad u \in E_n, n \in \mathbb{N}_0. \tag{4.2}$$

Indeed, it is enough to set

$$\begin{cases} |\cdot|_0 = \|\cdot\|_0, \\ |\cdot|_{n+1} = \frac{1}{M_1 \dots M_n}\|\cdot\|_{n+1}, \quad n \in \mathbb{N}. \end{cases} \tag{4.3}$$

Therefore in the sequel of this section we always assume the monotonicity of the sequence of norms $\|\cdot\|_n$.

On the vector space $E^+ = \bigcup_{n\in\mathbb{N}_0} E_n$ the locally convex inductive topology $\mathcal{T}_{\text{ind}}$ is imposed; a balanced convex U in E^+ is a neighbourhood of zero in $(E^+, \mathcal{T}_{\text{ind}})$ if $U \cap E_n$ is a neighborhood of zero for all $n \in \mathbb{N}_0$. In general, topological properties of a subset G in E^+ cannot simply be reduced to the corresponding properties of the set $G \cap E_n$ in E_n $(n \in \mathbb{N}_0)$. For instance, Б. М. Макаров [30], p.172, has given various examples of inductive limits in which bounded set in $(E^+, \mathcal{T}_{\text{ind}})$ is not bounded in any of the spaces $(E_n, \|\cdot\|_n)$, or even not situated in any of them. However, there exists a condition which ensures that a set G is contained and bounded in $(E^+, \mathcal{T}_{\text{ind}})$ if it is bounded in some E_{n_0} (n_0 depends on G); such a sequence $\{E_n \colon n \in \mathbb{N}_0\}$ or its inductive limit E^+ is said to be *regular*.

Example 4.1.1 (Пример 1 in [30]). Let X_n be the space of all sequences $(x_{i,j})_{i,j\in\mathbb{N}}$ fulfilling the conditions:

(a) $\forall_{i\in\{n+1,n+2,\dots\}}\ \exists_{a_i\in\mathbb{R}}\ \lim\limits_{j\to\infty} x_{i,j} = a_i$.

(b) $\lim\limits_{j\to\infty} \frac{x_{i,j}}{1+j} = 0, \quad i \in \{1,2,\dots,n\}$.

We can define the norm in the space X_n by

$$\|x\|_{X_n} = \|x\|_n = \sup_{i,j\in\mathbb{N}} \frac{|x_{i,j}|}{e^n_{ij}}, \quad \text{where} \quad e^n_{ij} = \begin{cases} 1+j & \text{for } i \le n, \\ 1 & \text{for } i > n. \end{cases}$$

It is easy to see that $(X_n, \|\cdot\|_n)$ is a separable Banach space for all $n \in \mathbb{N}$ and

$$X_n \subseteq X_{n+1}, \quad n \in \mathbb{N}.$$

Let $X^+ = \text{ind}_{n\to\infty} X_n$ be inductive limit of spaces X_n. Observe that the topology in $(X^+, \mathcal{T}_{\text{ind}})$ is not weaker than the topology coming from the norm $\|x\| = \sup\limits_{i,j\in\mathbb{N}} \frac{|x_{i,j}|}{1+j}$.

Consider the sequence $A = (X^{(k,n)})_{k,n\in\mathbb{N}}$ where

$$X^{(k,n)} = (X^{(k,n)}_{i,j})_{i,j\in\mathbb{N}}, \quad \text{where } X^{(k,n)}_{i,j} = \begin{cases} -1 & \text{while } j = 2m, i \le n, m \le k, \\ 1 & \text{in anothers cases.} \end{cases}$$

Observe that $A \subseteq X_1$ and $\|X^{(k,n)}\|_1 = 1$ $(k, n \in \mathbb{N})$, so A is bounded in X_1 and is bounded in X^+. Let

$$y^{(n)} = (y_{ij}^{(n)})_{i,j\in\mathbb{N}} \quad y_{ij}^{(n)} = \begin{cases} -1 & \text{while } j = 2m, i \leq n, \\ 1 & \text{in anothers cases.} \end{cases}$$

We have $y^{(n)} \in X_n$ and taking into consideration that $\lim\limits_{j\to\infty} y_{nj}^{(n)}$ does not exist, we can see $y^{(n)} \notin X_{n-1}$. Next we have $\|X^{(n,k)} - y^{(n)}\| = \frac{2}{2k+3} \xrightarrow[k\to\infty]{} 0$. Thus $X^{(n,k)} \xrightarrow[k\to\infty]{} y^{(n)}$ in X^+ and the sequence $B = (y^{(n)})_{n\in\mathbb{N}}$ is contained in a closure of the bounded sequence A in X^+ $(B \subseteq \mathrm{cl}_{X^+}A)$, therefore bounded in X^+. Whereas $y^{(n)} \notin X_{n-1}$, sequence B is not included in none of the spaces space X_n. Thus the inductive limit X^+ is not regular.

Theorem 4.1.2 (Theorem in [14] and Theorem I.1.1 in [19]). *Let K_n be the unit closed ball in E_n. If for all sequences $\{\varepsilon_m \colon m \in \mathbb{N}_0\}$ of positive numbers and for all $n \in \mathbb{N}_0$, sum $\sum\limits_{m=0}^{n} \varepsilon_m K_m$ is closed in E_{n+1}, then the inductive limit $E^+ = \mathrm{ind}_{n\to\infty} E_n$ is regular.*

Proof. Let $A \subseteq E^+$ be $\mathcal{T}$-bounded and not $\mathcal{T}_n$-bounded for all $n \in \mathbb{N}$. Assume that for all $n \in \mathbb{N}$ there are $\varepsilon_m > 0$ and $x_m \in A$, $m \in \{1, \ldots, n\}$ with

$$\frac{1}{m} x_m \notin \sum_{i=1}^{n} \varepsilon_i K_i =: U_n, \quad m \in \{1, \ldots, n\}. \tag{4.4}$$

Then, because $U_n \subseteq U_{n+1}$, (4.4) holds for all $m, n \in \mathbb{N}$:

$$\frac{1}{m} x_m \notin U := \bigcup_{n=1}^{\infty} U_n, \quad m \in \mathbb{N}.$$

A nonempty $\mathcal{T}$-neighborhood of zero is contained in U and $\left(\frac{1}{m} x_m\right)_{m\in\mathbb{N}}$ $\mathcal{T}$-converges to zero (A being $\mathcal{T}$-bounded). A contradiction is established. Thus (ε_m) and (x_m) with (4.4) will be constructed: $A \not\subseteq K_1$, so there is an $x_1 \in A \setminus K_1$, $\varepsilon_1 := 1$. Proceeding by induction, assume that $n \in \mathbb{N}$, $\varepsilon_m > 0$ and $x_m \in A$ with

$$\frac{1}{m} x_m \notin \sum_{i=1}^{n} \varepsilon_i K_i =: U_n, \quad m \leq n$$

are given. By the assumption of the theorem, U_n is closed in E_{n+1}; then $\frac{1}{m}x_m \in E_{n+1} \setminus U_n$, which is open. Therefore, there exists an opened neighborhood V with $\frac{1}{m}x_m - V \subseteq E_{n+1} \setminus U_n$ and there exists ε_{m+1} such that $\varepsilon_{m+1}K_{n+1} \subseteq V$, thus

$$\frac{1}{m}x_m \notin U_n + \varepsilon_{n+1}K_{n+1} =: U_{n+1}, \quad m \leq n$$

(x_m need not be elements of E_{n+1}). Since U_{n+1} is bounded in E_{n+1}, the set A is not contained in $(n+1)U_{n+1}$, what implies the existence of an $x_{n+1} \in A$ with

$$\frac{1}{n+1}x_{n+1} \notin U_{n+1} = \sum_{i=1}^{n+1} \varepsilon_i K_i. \qquad \square$$

As a consequence of the idem theorem we have the following corollaries.

Corollary 4.1.3 (Corollary 1 in [14] and Corollary I.1.1 in [19]). *If there exists a semireflexive locally convex space F and an injective continuous operator $T\colon E^+ = \mathrm{ind}_{n\to\infty}E_n \to F$ such that TK_n is closed for all $n \in \mathbb{N}_0$, then space E^+ is regular.*

Proof. All TK_n are $\sigma(F, F^*)$-compact and because of

$$\sum_{m=1}^{n} \varepsilon_m TK_m = T\left(\sum_{m=1}^{n} \varepsilon_m K_m\right),$$

they are closed in particular: T being injective and continuous yields that the application of the Theorem 4.1.2 applies. $\square$

Corollary 4.1.4 (Corollary 2 in [14] and Corollary I.1.2 in [19]). *Let $(F_n, \|\cdot\|_n)$ be a sequence of reflexive Banach spaces such that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$ and for all $n \in \mathbb{N}$*

$$\|u\|_n \leq \|u\|_{n+1}, \quad u \in F_n.$$

Then the inductive limit of the sequence of dual spaces $E_n = F_n'$ is regular.

Proof. Under our assumption E_n is a Banach space with the norm $\|\cdot\|_n'$ such that $E_n \subseteq E_{n+1}$ and for all $n \in \mathbb{N}$

$$\|u\|_{n+1}' \leq \|u\|_n', \quad u \in E_n.$$

The unit ball K_n in E_n is $\sigma(E_n, F_n)$-compact and the inclusion map is $\sigma(E_n, F_n) \to \sigma(E_{n+1}, F_{n+1})$ continuous, so K_n is $\sigma(E_{n+1}, F_{n+1})$-compact and closed. $\square$

As a consequence of Corollary 4.1.4 we can prove what follows:

Corollary 4.1.5. *Inductive limit of a sequence of locally convex spaces with weakly compact inclusion is regular.*

Proof. Proof of this corollary is based on previous Floret's Corollary and on the Grothendieck's Lemma:

Lemma 4.1.6. *Assuming K be a balanced, convex, weakly compact subset of a locally convex space E, then there is a Banach space B such that* $\operatorname{lin}(K) = B'$ *isometrically and* $\operatorname{lin}(K) = B' \hookrightarrow E$ *is* $\sigma(B', B) \to \sigma(E, E')$ *continuous.*

In this Lemma $\operatorname{lin}(K)$ denotes linear hull of a set K equipped with the Minkowski norm m_A. □

Another proof of this corollary, not based on the results of K. Floret, can be found among others in H. Komatsu [25], Theorem 3, p.372.

If the inductive limit $E^+ = \operatorname{ind}_{n\to\infty} E_n$ is regular, then it easily follows that E^+ is bornological and barreled. In addition, if an interpolation-type inequality is satisfied, then we can also characterize converging nets or sequences, Cauchy nets or sequences and compact sets in E^+ and consequently obtain the completeness of E^+.

Theorem 4.1.7 (Theorem I.1.4 in [19]). *Suppose that the inductive limit E^+ is regular and that E^+ is continuously embedded in some Banach space $(E, \|\cdot\|)$. Assume that for each $n \in \mathbb{N}_0$ there is $k > n$ and a function $\phi_{n,k}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that the interpolation inequality*

$$\|u\|_k \le \phi_{n,k}(\|u\|_n, \|u\|), \quad u \in E_n \tag{4.5}$$

holds. Additionally assume function $\phi_{n,k}$ is monotone in each of its variables and such that $\phi_{n,k}(t,s) \to 0$ as $s \to 0$ for each fixed t. Then we have

(i) *A set $\{u_\alpha\colon \alpha \in I\}$ in E^+ converges (to zero) in E^+ if and only if it converges (to zero) in some E_n.*
(ii) *A set $\{u_\alpha\colon \alpha \in I\}$ in E^+ is a Cauchy net in E^+ if and only if it is a Cauchy net in some E_n.*
(iii) *A subset G in E^+ is compact if and only if it is compact in some E_n (the same is true for relative compactness.)*
(iv) *E^+ is complete.*

Proof. (i) Assume that a set $\{u_\alpha : \alpha \in I\}$ converges to zero in E^+. Then it is bounded in E^+ and the regularity of inductive limit E^+ implies the existence of some $n \in \mathbb{N}_0$ such that $\{u_\alpha : \alpha \in I\} \subseteq E_n$ and $\|u_\alpha\| \leq M_n$ for all $\alpha \in I$ where M_n is a positive constant. On the other hand, the continuity of embedding from E^+ into E ensures that the net converges to zero in E, i.e. $\|u_\alpha\| \to 0$. Then, by assumption, there exists some $k > n$ and a function $\phi_{n,k}$ such that (4.5) is satisfied. In particular

$$\|u_\alpha\|_k \leq \phi_{n,k}(\|u_\alpha\|_n, \|u_\alpha\|) \leq \phi_{n,k}(M_n, \|u_\alpha\|),$$

from which follows that $\|u_\alpha\| \to 0$. The converse is trivial.

(ii) The proof is entirely similar to that of (i) and is omitted.

(iii) Let the subset G be compact in E^+. In particular it is bounded in E^+ and the regularity of inductive limit implies the existence of some $n \in \mathbb{N}_0$ such that $G \subseteq E_n$ and $\|u\| \leq M_n$ for all $u \in G$, where M_n is a positive constant. By our assumption we can choose some $k > n$ such that (4.5) is valid. Then for a given sequence $\{u_m : m \in I\} \subseteq G$, the compactness of G in E^+ implies the existence of a subsequence $\{u_{m'}\} \to v \in G$ in E^+ and hence in E, i.e. $\|u_{m'} - v\| \to 0$ as $m' \to \infty$. From

$$\|u_{m'} - v\|_k \leq \phi_{n,k}(\|u_{m'} - v\|_n, \|u_{m'} - v\|) \leq \phi_{n,k}(2M_n, \|u_{m'} - v\|),$$

can be followed directly that $\{u_{m'}\} \to v$ in E_k. This shows the sequential compactness of G in E_k, which is however equivalent to the compactness in the Banach space E_k. The converse is trivial.

(iv) Take a Cauchy sequence $\{u_n : n \in \mathbb{N}\}$ in E^+. Then, from (ii) it is a Cauchy sequence in E_n for some $n \in \mathbb{N}$ and, because E_n is a Banach space, converges in E_n. Applying (i), we have that $\{u_n : n \in \mathbb{N}\}$ converges in E^+. □

Remark 4.1.8. In the papers of K. Floret [15] and U. Sztaba, Wl. Kierat [22], the authors introduce another way to construct inductive limits. They assume that spaces in the sequence are locally convex linear-topological Hausdorff spaces. They prove additionally that the inductive limit can be complete under weaker assumptions than ours.

In analysis we meet projective limits as well as inductive limits of Banach spaces. However, the theory of projective limits of Banach spaces is much simpler and more "classical" than that for inductive limits. For completeness and easier citation we state the following standard results on projective limits of Banach spaces, the proof of which is straightforward and can be found in standard textbooks on functional analysis and generalized functions (for example I. M. Gel'fand, et al. [16]).

Let there be given a sequence of Banach spaces $\{(F_n, \|\cdot\|_n)\colon n \in \mathbb{N}_0\}$ such that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}_0$ and let all the embeddings be continuous, i.e. assume that for each $n \in \mathbb{N}_0$ there exists $M_n > 0$

$$\|u\|_n \leq M_n \|u\|_{n+1}, \quad u \in E_n. \tag{4.6}$$

If we set

$$\begin{cases} |\cdot|_0 = \|\cdot\|_0, \\ |\cdot|_{n+1} = M_1 \dots M_n \|\cdot\|_{n+1}, \quad n \in \mathbb{N}_0, \end{cases} \tag{4.7}$$

then each of the new norms $|\cdot|_n$ on F_n is equivalent to the original $\|\cdot\|_n$ and they are monotone increasing:

$$|\cdot|_n \leq |\cdot|_{n+1}, \quad n \in \mathbb{N}_0. \tag{4.8}$$

Let us assume $F^- = \bigcap\limits_{n\in\mathbb{N}_0} F_n$ to be non-empty and equip F^- with locally convex topology $\mathcal{T}_{\text{proj}}$ generated by the sequence of norms $\{\|\cdot\|_n\colon n \in \mathbb{N}_0\}$. A neighbourhood of zero $U_{p,\varepsilon}$ is defined by a positive integer p and $\varepsilon > 0$, and consists of all $u \in F^-$ which satisfy the p inequalities $\|u\|_1 < \varepsilon$, $\|u\|_2 < \varepsilon$, $\dots$, $\|u\|_p < \varepsilon$, which is equal to the set of all $u \in F^-$ satisfying $\|u\|_p$. We can introduce a metric in space F^- by

$$\rho(u,v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|u-v\|_n}{1+\|u-v\|_n}$$

The distance ρ is invariant with respect to a translation and defines a topology in F^-, which is identical with the original topology.

Proof. First, take $U_{n,\eta}$ with $n \in \mathbb{N}$ and $\eta > 0$. Take $\varepsilon \in (0,1)$ such that $\frac{\varepsilon}{1-\varepsilon} < \eta$ and $r < \frac{\varepsilon}{2^n}$. We will show that $K_\rho(0,r) \subseteq U_{n,\eta}$. Take $u \in K_\rho(0,r)$, then $\frac{1}{2^n}\frac{\|u\|_n}{1+\|u\|_n} < r < \frac{\varepsilon}{2^n}$ and $\|u\|_n < \frac{\varepsilon}{1-\varepsilon} < \eta$.
Next, take $r > 0$. Then, because of the sequence $\frac{1}{2^n} \to 0$, there exists $k \in \mathbb{N}$ such that $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^k} > \frac{r}{2}$. We have $\sum\limits_{i=k+1}^{\infty} \frac{1}{2^i} \leq \frac{r}{2}$. Take $u \in U_{k,\frac{r}{2}}$. Then $\rho(0,u) \leq \sum\limits_{i=1}^{k} \frac{1}{2^i}\frac{\|u\|_i}{\|u\|_i+1} + \sum\limits_{i=k+1}^{\infty} \frac{1}{2^i} < \sum\limits_{i=1}^{k} \frac{1}{2^i}\frac{r}{2} + \frac{r}{2} \leq r$. This shows that $U_{k,\frac{r}{2}} \subseteq K_\rho(0,r)$. $\square$

Then we conclude:

Theorem 4.1.9 (Theorem I.1.5 in [19]).

(i) *A sequence* $\{u_n : n \in \mathbb{N}_0\}$ *in* F^- *converges (to zero) in* $(F^-, \mathcal{T}_{\text{proj}})$ *if and only if it converges (to zero) in all the spaces* F_n.
(ii) *A sequence* $\{u_n : n \in \mathbb{N}_0\}$ *in* F^- *is a Cauchy sequence in* $(F^-, \mathcal{T}_{\text{proj}})$ *if and only if it is a Cauchy sequence in each of the spaces* F_n.
(iii) *A set* G *in* F^- *is compact in* $(F^-, \mathcal{T}_{\text{proj}})$ *if and only if it is compact in each of the spaces* F_n. *The same applies to relative compactness.*
(iv) F^- *is a Frechét space.*

Proof. (iv) Suppose that $(u_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in F^-. From (ii), $(u_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in each F_n and has a limit $u^{(n)}$ in each of these spaces. Within the view of the $F_n \subseteq F_{n-1}$, all of the elements $u^{(n)}$ $(n \in \mathbb{N})$ are, in essence, the same element, which, therefore, belongs to each of the F_n, so belongs to F^-. Let us denote this element of F^- by u. Since $\|u_m - u\|_n \xrightarrow[m\to\infty]{} 0$ for every $n \in \mathbb{N}$ we have $u = \lim_{n\to\infty} u_n$ in the topology of F^-. Thus, F^- is a complete space. □

Now we proceed to characterize the continuous mappings between spaces which are inductive limits or projective limits of sequences in Banach spaces.

Theorem 4.1.10 (Theorem I.1.6 in [19]). *Let* $\{(E_n, \|\cdot\|_{E_n}) : n \in \mathbb{N}_0\}$ *and* $\{(\tilde{E}_n, \|\cdot\|_{\tilde{E}_n}) : n \in \mathbb{N}_0\}$ *be two sequences of Banach spaces with inductive limits* E^+ *and* $\tilde{E}^+$ *respectively, and let* $\tilde{E}^+$ *satisfy the condition in Theorem 4.1.7. Let* $\{(F_n, \|\cdot\|_{F_n}) : n \in \mathbb{N}_0\}$ *and* $\{(\tilde{F}_n, \|\cdot\|_{\tilde{F}_n}) : n \in \mathbb{N}_0\}$ *be two sequences of Banach spaces with nonempty projective limits* F^- *and* $\tilde{F}^-$ *respectively.*

(a) *For a linear mapping* $T : E^+ \to \tilde{E}^+$ *the following conditions are equivalent:*
   (i) T *is continuous.*
   (ii) *If a sequence* $\{u_m : m \in \mathbb{N}_0\}$ *is contained and converges (to zero) in* E_n *for some* $n \in \mathbb{N}_0$, *then* $\{Tu_m : m \in \mathbb{N}_0\}$ *is contained and converges (to zero) in some* $\tilde{E}_{\tilde{n}}$.
   (iii) *For any* $n \in \mathbb{N}_0$ *there exists an* $\tilde{n} \in \mathbb{N}_0$ *and a constant* $M_{n,\tilde{n}} > 0$ *such that*

$$\|Tu\|_{\tilde{E}_{\tilde{n}}} \leq M_{n,\tilde{n}} \|u\|_{E_n}, \quad u \in E_n. \tag{4.9}$$

(b) *For a linear mapping* $T : F^- \to \tilde{F}^-$ *the following conditions are equivalent:*

(i) *T is continuous.*
(ii) *If a sequence $\{u_m : m \in \mathbb{N}_0\}$ converges (to zero) in each F_n, then $\{Tu_m : m \in \mathbb{N}_0\}$ converges (to zero) in each $\tilde{F}_{\tilde{n}}$.*
(iii) *For any $\tilde{n} \in \mathbb{N}_0$ there exists an $n \in \mathbb{N}_0$ and a constant $M_{n,\tilde{n}} > 0$ such that*

$$\|Tu\|_{\tilde{F}_{\tilde{n}}} \le M_{n,\tilde{n}} \|u\|_{F_n}, \quad u \in F^-.$$

(c) *For a linear mapping $T\colon E^+ \to F^-$ the following conditions are equivalent:*
(i) *T is continuous.*
(ii) *If a sequence $\{u_m : m \in \mathbb{N}_0\}$ is contained and converges (to zero) in E_n for some $n \in \mathbb{N}_0$, then $\{Tu_m : m \in \mathbb{N}_0\}$ converges (to zero) in all the spaces F_k $(k \in \mathbb{N}_0)$.*
(iii) *For each $n, k \in \mathbb{N}_0$ there exists a constant $M_{n,k} > 0$ such that*

$$\|Tu\|_{F_k} \le M_{n,k} \|u\|_{E_n}, \quad u \in E_n.$$

(d) *For a linear mapping $T\colon F^- \to E^+$ the following conditions are equivalent:*
(i) *T is continuous.*
(ii) *If a sequence $\{u_m : m \in \mathbb{N}_0\}$ converges (to zero) in all the spaces F_n $(n \in \mathbb{N}_0)$, then $\{Tu_m : m \in \mathbb{N}_0\}$ is contained and converges (to zero) in some E_k.*
(iii) *There exists a $n, k \in \mathbb{N}_0$ and a constant $M_{n,k} > 0$ such that*

$$\|Tu\|_{E_k} \le M_{n,k} \|u\|_{F_n}, \quad u \in F^-. \tag{4.10}$$

Proof. As in all the four cases the proofs are similar, we focus on the proofs of (a) and (d) only.

(a) First we prove the case of (a).
(iii) $\Rightarrow$ (ii) is trivial.
(ii) $\Rightarrow$ (iii) Suppose that there exists an $n \in \mathbb{N}_0$ such that

$$\sup_{\|u\|_{E_n}=1} \|Tu\|_{\tilde{E}_i} = \infty, \quad i \in \mathbb{N}_0.$$

Then, for each $i \in \mathbb{N}_0$ we can find a sequence $\{u_{ij} : j \in \mathbb{N}_0\}$ such that

$$\|u_{ij}\|_{E_n} = 1 \quad \text{and} \quad \|Tu_{ij}\|_{\tilde{E}_i} \ge j^2, \; j \in \mathbb{N}_0.$$

Form the diagonal sequences $\{Tu_{ii} : i \in \mathbb{N}_0\}$ and $\{u_{ii} : n \in \mathbb{N}_0\}$. We have for each $j \in \mathbb{N}_0$ and $i > j$

$$\|Tu_{ii}\|_{\tilde{E}_j} \ge \|Tu_{ii}\|_{\tilde{E}_i} \ge i^2$$

and consequently

$$\left\|T\left(\frac{u_{ii}}{i}\right)\right\|_{\tilde{E}_j} \to \infty \text{ as } i \to \infty \text{ for each } j \in \mathbb{N}_0.$$

However

$$\left\|\frac{u_{ii}}{i}\right\|_{E_n} = \frac{1}{i} \to 0 \text{ as } i \to \infty.$$

The two relations above contradict with the statement in (ii). Thus we have proved the equivalence of (ii) and (iii).

(i) $\Rightarrow$ (ii) If a sequence $\{u_m : m \in \mathbb{N}_0\}$ is contained in some E_n and converges to zero there, then it converges to zero in E^+ and from the continuity of T it follows that $\{Tu_m : m \in \mathbb{N}_0\}$ converges to zero in $\tilde{E}^+$. Theorem 4.1.7 (i) ensures the existence of some $\tilde{n} \in \mathbf{N}_0$ such that $\{Tu_m : m \in \mathbb{N}_0\}$ is contained, and converges to zero, in $\tilde{E}_{\tilde{n}}$.

(iii) $\Rightarrow$ (i) Let us consider the restriction of T to the subspace E_n ($n \in \mathbb{N}_0$). For a sequence $\{u_m : m \in \mathbb{N}_0\}$ lying and converging to zero in E_n, since there exists an $\tilde{n} \in \mathbb{N}_0$ such that (4.9) holds, $\{Tu_m : m \in \mathbb{N}_0\}$ converges to zero in $\tilde{E}_{\tilde{n}}$ and therefore in $\tilde{E}^+$. Thus all the restrictions are continuous. Now let U be a convex neighborhood of zero in $\tilde{E}^+$. Obviously $T^{-1}(U)$ is convex in E^+ and $E_n \cap T^{-1}(U) = (T|_{E_n})^{-1}(U)$ is a neighborhood of zero in E_n. So $T^{-1}(U)$ is a neighborhood of zero in E^+ and T is continuous.

(d) Now we prove the case (d).

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) Suppose that there is no pair of n and k such that (4.10) is valid. Then for any $n, k \in \mathbb{N}_0$ we can find a sequence $\{u_{n,k,j} : j \in \mathbb{N}_0\}$ in F^- such that

$$\|u_{n,k,j}\|_{F_n} = 1 \quad \text{and } \|Tu_{n,k,j}\|_{E_k} \geq j^2, \text{ for all } j \in \mathbb{N}_0.$$

Now we form a sequence $\{v_j : j \in \mathbb{N}_0\}$ with $v_j = u_{j,j,j}$. For each $n \in \mathbb{N}_0$ we have

$$\left\|\frac{v_j}{j}\right\|_{F_n} \leq \left\|\frac{v_j}{j}\right\|_{F_j} = \frac{1}{j}, \quad \text{for all } j \geq n.$$

For each $k \in \mathbb{N}_0$ we have

$$\left\|T\left(\frac{v_j}{j}\right)\right\|_{E_k} \geq \left\|T\left(\frac{v_j}{j}\right)\right\|_{E_j} \geq j \quad \text{for all } j \geq n.$$

The above readily lead to the conclusion that

$$\left\|\frac{v_j}{j}\right\|_{F_n} \xrightarrow[j\to\infty]{} 0 \quad \text{for all } n \in \mathbb{N}_0$$
$$\left\|T\left(\frac{v_j}{j}\right)\right\|_{E_k} \xrightarrow[j\to\infty]{} \infty \quad \text{for all } k \in \mathbb{N}_0$$

which is a contradiction to the statement in (ii).

(iii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (ii) follows from the Theorem 4.1.7 (i). □

Corollary 4.1.11 (Corollary I.1.7 in [19]). *In each of the four cases in the Theorem above, a linear operator T is continuous iff it is bounded, i.e. it maps bounded subset into bounded subset.*

Corollary 4.1.12 (Corollary I.1.8 in [19]).

$$(E^+)^* = \bigcap_{n\in\mathbb{N}_0} E_n^*, \quad (F^-)^* = \bigcup_{n\in\mathbb{N}_0} F_n^-.$$

Corollary 4.1.13 (Corollary I.1.9 in [19]).

(a) *If $E^+ = \tilde{E}^+$ are vector spaces, then the following are equivalent:*
  (i) *$E^+ = \tilde{E}^+$ as topological vector spaces and both E^+, $\tilde{E}^+$ satisfy the mentioned condition in Theorem* 4.1.10.
  (ii) *If a sequence $\{u_m : m \in \mathbb{N}_0\}$ is contained and converges (to zero) in some E_n so it does in some $\tilde{E}_{\tilde{n}}$ and vice versa.*
  (iii) *For any $n \in \mathbb{N}_0$ there exists some $\tilde{n} \in \mathbb{N}_0$ and a positive constant $M_{n,\tilde{n}}$ such that T maps E_n into $\tilde{E}_{\tilde{n}}$ and*

$$\|u\|_{\tilde{E}_{\tilde{n}}} \le M_{n,\tilde{n}} \|u\|_{E_n}, \quad u \in E_n,$$

and vice versa.

(b) *If $F^- = \tilde{F}^-$ are vector spaces, then the following are equivalent:*
  (i) *$F^- = \tilde{F}^-$ as topological vector spaces.*
  (ii) *If a sequence $\{u_m : m \in \mathbb{N}_0\}$ converges (to zero) in each of the spaces F_n $(n \in \mathbb{N}_0)$, then it is convergent in each of the spaces $\tilde{F}_{\tilde{n}}$ $(\tilde{n} \in \mathbb{N}_0)$.*
  (iii) *For any $\tilde{n} \in \mathbb{N}_0$ there exists some $n \in \mathbb{N}_0$ and a positive constant $M_{n,\tilde{n}}$ such that*

$$\|u\|_{\tilde{F}_{\tilde{n}}} \le M_{n,\tilde{n}} \|u\|_{F_n}, \quad u \in F^+,$$

and vice versa.

4.2 Regular Spaces and Hyper-spaces

Let $(X, \|\cdot\|)$ be a Banach space and let B be a linear operator of the type $(\omega, M(\theta))$ in X which resolvent contains 0.

Definition 4.2.1 (Definition I.2.1 in [19]). Let $\sigma \in (0, \infty)$.

1. $X_B^\sigma = (D(B^\sigma), \|\cdot\|_{\sigma,B})$, where $\|u\|_{\sigma,B} = \|B^\sigma u\|$ for $u \in D(B^\sigma)$; $X_B^0 = (X, \|\cdot\|)$.
   $\|\cdot\|_{\sigma,B}$ is often abbreviated as $\|\cdot\|_\sigma$.
2. $X_B^{-\sigma}$ is the completion of $(X, \|\cdot\|_{-\sigma,B})$, where

$$\|u\|_{-\sigma,B} = \|u\|_{-\sigma} = \|B^{-\sigma}u\|$$

   for $u \in X$.

Proposition 4.2.2 (Proposition I.2.2 in [19]). *The scale of spaces $\{X_B^\sigma : \sigma \in \mathbb{R}\}$ is a scale of Banach spaces. For any $\tau > \sigma > 0$ we have the relation*

$$X_B^\tau \subseteq X_B^\sigma \subseteq X \subseteq X_B^{-\sigma} \subseteq X_B^{-\tau}$$

where every smaller space is densely and continuously embedded into any bigger one.

Having a scale of Banach spaces $\{X_B^\sigma : \sigma \in (-\infty, +\infty)\}$ we can construct its inductive limits and projective limits.

Definition 4.2.3 (Definition I.2.3 in [19]).

1. For $\sigma \in (-\infty, \infty]$, $X_B^{\sigma-} = \bigcap\limits_{\tau<\sigma} X_B^\tau$ with a projective limit topology; $X_B^{\infty-} = X_B^\infty$.
2. For $\sigma \in [-\infty, \infty)$, $X_B^{\sigma+} = \bigcup\limits_{\tau>\sigma} X_B^\tau$ with a inductive limit topology; $X_B^{(-\infty)+} = X_B^{-\infty}$.

We shall remark, because of Proposition 4.2.2 above, the inductive limits and projective limits are well defined for any sequence of Banach spaces with monotonic indices converging to their righthand limits.

Proposition 4.2.4 (Proposition I.2.4 in [19]). *Among the Banach spaces $\{X_B^\sigma : \sigma \in \mathbb{R}\}$, the inductive limits $\{X_B^{\sigma+} : \sigma \in [-\infty, \infty)\}$ and projective limits $\{X_B^{\sigma-} : \sigma \in (-\infty, \infty]\}$, the following relations hold* $(0 < \sigma < \tau < \infty)$

$$X_B^\infty \subseteq X_B^{\tau+} \subseteq X_B^\tau \subseteq X_B^{\tau-} \subseteq X_B^{\sigma+} \subseteq X_B^\sigma \subseteq X_B^{\sigma-} \subseteq X_B^{0+} \subseteq X_B$$
$$X_B \subseteq X_B^{0-} \subseteq X_B^{-\sigma+} \subseteq X_B^{-\sigma} \subseteq X_B^{-\sigma-} \subseteq X_B^{-\tau+} \subseteq X_B^{-\tau} \subseteq X_B^{-\tau-} \subseteq X_B^{-\infty}.$$

In this case each smaller space is densely and continuously embedded into a bigger one except for X_B^{∞}.

We omit a proof of this Proposition again only mentioning that X_B^{τ} ($\tau > 0$) is known to be dense in $X_B^{\tau-}$ via the same procedure as we used to prove that $D(B^{\tau})$ is a core for B^{σ} ($0 < \sigma < \tau$).
The next two theorems clarify the topological properties of these spaces of inductive limits and projective limits.

Definition 4.2.5. A *barreled set* in a topological vector space is a set which is convex, balanced, absorbing and closed. A *barreled space* is Hausdorff topological vector space where every barreled set in the space is a neighbourhood of zero. A *Montel space* is a barreled topological vector space where every closed and bounded set is compact.

Theorem 4.2.6 (Theorem I.2.5 in [19]). *Assume that either the Banach space is reflexive or all the operators $B^{-\sigma}$ ($\sigma > 0$) are compact in X. Then we have*

1. *All the inductive limit spaces $X_B^{\sigma+}$ ($\sigma \in [-\infty, +\infty)$) are regular.*
2. *For any $\sigma \in \mathbb{R}$ a bounded net $\{u_\alpha : \alpha \in I\}$ in $X_B^{\sigma+}$ converges (to zero) if and only if it converges (to zero) in some X_B^{τ}, ($\tau > \sigma$).*
3. *For any $\sigma \in \mathbb{R}$ a bounded net $\{u_\alpha : \alpha \in I\}$ in $X_B^{\sigma+}$ is a Cauchy net if and only if it is a Cauchy net in some X_B^{τ}, ($\tau > \sigma$). All the spaces $X_B^{\sigma+}$ ($\sigma \in \mathbb{R}$) are complete.*
4. *For any $\sigma \in \mathbb{R}$ a set G in $X_B^{\sigma+}$ is compact if and only if it is a compact set in some X_B^{τ} ($\tau > \sigma$). The same is true for relative compactness.*
5. *Each of the spaces $X_B^{\sigma+}$ ($\sigma \in \mathbb{R}$) is barreled and bornological; it is Montel if and only if all the mappings $B^{-\tau} : X \to X$ ($\tau > 0$) are compact.*

Proof. It is readily seen that for each $\sigma > 0$ the isometry operator $B^{-\sigma} : X \to X_B^{\sigma}$ extends uniquely to an isometry operator from $X_B^{-\sigma}$ onto X still denoted by $B^{-\sigma}$; its inverse extends $B^{\sigma} : X_B^{\sigma} \to X$ and is denoted by B^{σ} again. In this way, via X, we have an isometric operator from X_B^{σ} onto X_B^{τ}, denoted by $B^{\sigma-\tau}$, for each $\sigma, \tau \in \mathbb{R}$.

(1). If the Banach space is reflexive, so is each X_B^{σ} ($\sigma \in \mathbb{R}$). It is also evident that $B^{-\tau} : X \to X$ ($\tau > 0$) is compact iff the inclusion mapping from X_B^{σ} into $X_B^{\sigma-\tau}$ is compact. Then the regularity for each of the inductive limit space $X_B^{\sigma+}$ ($\sigma \in [-\infty, \infty)$) follows from Corollary 4.1.4 and the remarks following it.

(2), (3), (4) are consequences of (1) and Theorem 4.1.7 since now we have an interpolation inequality

$$\|u\|_\tau \le C(\sigma,\tau,\theta)\|u\|_\sigma^{\frac{\theta-\tau}{\theta-\sigma}}\|u\|_\theta^{\frac{\tau-\sigma}{\theta-\sigma}} \quad (\sigma<\tau<\theta)$$

which is just a corresponding property for fractional powers (see Theorem 1.0.7). Here for fixed σ we apply $E = X_B^\sigma$ in Theorem 4.1.7.

$X_B^{\sigma+}$ is barreled and bornological since it is regular. If all the mappings $B^{-\tau}\colon X \to X$ $(\tau > 0)$ are compact, then, equivalently, all the inclusion mappings $i\colon X_B^\theta \to X_B^{\theta-\tau}$ $(\theta \in \mathbb{R})$ are compact. Thus, if G is bounded and closed in $X_B^{\sigma+}$, then by (1) there exists $\theta > \sigma$ such that G is bounded in X_B^θ. Hence it is compact in $X_B^{\frac{\theta+\sigma}{2}}$, and so it is in $X_B^{\sigma+}$. This shows that $X_B^{\sigma+}$ is Montel. Conversely, assume that $X_B^{\sigma+}$ is Montel for some $\sigma \in \mathbb{R}$. For $\tau > 0$ and a bounded set G in X, $B^{-(\sigma+\tau)}G$ is bounded is $X_B^{\sigma+\tau}$, and so it is in $X_B^{\sigma+}$. Since $X_B^{\sigma+}$ is Montel, $B^{-(\sigma+\tau)}G$ is relatively compact in $X_G^{\sigma+}$. By (4) above there exists some $\delta \in (0,1)$ such that $B^{-(\sigma+\tau)}G$ is relatively compact in $X_B^{\sigma+\delta\tau}$. This in turn is equivalent to $B^{\sigma+\delta\tau}B^{-(\sigma+\tau)}G = B^{-(1-\delta)\tau}G$ being relatively compact in X, and so is $B^{-\tau}G = B^{-\delta}B^{-(1-\delta)\tau}G$. This proves the compactness of $B^{-\tau}$ for each $\tau > 0$. □

Theorem 4.2.7 (Theorem I.2.7 in [19]). *Let $\sigma \in (-\infty,+\infty]$. Then we have:*

1. *$X_B^{\sigma-}$ is a Frechét space.*
2. *A sequence in $X_B^{\sigma-}$ converges (to zero) if and only if it converges (to zero) in each of the spaces X_B^τ $(\tau < \sigma)$.*
3. *A sequence in $X_B^{\sigma-}$ is a Cauchy sequence if and only if it is a Cauchy sequence in each of the spaces X_B^τ $(\tau < \sigma)$.*
4. *A set G in $X_B^{\sigma-}$ is bounded (compact) if and only if it is bounded (compact) in each of the spaces X_B^τ $(\tau < \sigma)$. The same applies to relative compactness.*
5. *$X_B^{\sigma-}$ is Montel if and only if all the mappings $B^{-\tau}\colon X \to X$ $(\tau > 0)$ are compact.*

If the Banach space $(X, \|\cdot\|)$ and the operator B are replaced by the dual space $(X^*, \|\cdot\|_*)$ and the dual operator B^*, then we obtain another scale of Banach spaces $(X^*)_{B^*}^\sigma$ $(\sigma \in \mathbb{R})$ and their inductive limits $(X^*)_{B^*}^{\sigma+}$ $(\sigma \in [-\infty,\infty))$ and projective limits $(X^*)_{B^*}^{\sigma-}$ $(\sigma \in (-\infty,\infty])$. The norm of $(X^*)_{B^*}^\sigma$ is denoted by $\|\cdot\|_{*,B,\sigma}$, sometimes abbreviated to $\|\cdot\|_{*,\sigma}$. There is a natural duality relation between the two scales of spaces.

Theorem 4.2.8 (Theorem I.2.8 in [19]).
$(X^\sigma_B)^* = (X^*)^{-\sigma}_{B^*}$ and $X^\sigma_B \hookrightarrow [(X^*)^{-\sigma}_{B^*}]^*$ ($\sigma > 0$) isometrically via the duality pairing $\langle\cdot,\cdot\rangle_\sigma \colon X^\sigma_B \times (X^*)^{-\sigma}_{B^*} \to \mathbb{C}$

$$\langle u, f\rangle_\sigma = (B^\sigma u, (B^*)^{-\sigma} f), \quad u \in X^\sigma_B, f \in (X^*)^{-\sigma}_{B^*} \tag{4.11}$$

where $(\cdot,\cdot)$ is the duality pairing between X and X^*. If, furthermore, the space X is reflexive, then $X^\sigma_B = [(X^*)^{-\sigma}_{B^*}]^*$ isometrically by the same duality pairing (4.11).

Proof. For $u \in X^\sigma_B$ and $f \in (X^*)^{-\sigma}_{B^*}$ we have

$$|\langle u, f\rangle_\sigma| = |(B^\sigma u, (B^*)^{-\sigma} f)| \leq \|u\|_\sigma \|f\|_{*,-\sigma}. \tag{4.12}$$

Now let $f \in (X^*)^{-\sigma}_{B^*}$ be given and set $g = (B^*)^{-\sigma} f \in X^*$. There exists a sequence $\{\nu_n\} \subseteq X$ such that $\|\nu_n\| = 1$ and $|(\nu_n, g)| \to \|g\|_*$. Putting $u_n = B^{-\sigma}\nu_n$, then $u_n \in X^\sigma_B$, $\|u_n\|_\sigma = 1$ and

$$|\langle u_n, f\rangle_\sigma| = |(\nu_n, g)| \to \|g\|_* = \|f\|_{*,-\sigma}. \tag{4.13}$$

Then (4.12) and (4.13) together imply that $F = \langle\cdot, f\rangle_\sigma \colon X^\sigma_B \to \mathbb{C}$ belongs to $(X^\sigma_B)^*$ and $\|F\| = \|f\|_{*,-\sigma}$.

Conversely, if $F \colon X^\sigma_B \to \mathbb{C}$ is in $(X^\sigma_B)^*$, then, since X^σ_B and X are isometric to each other under the mapping B^σ, there exists a unique $g \in X^*$ such that $F(u) = (B^\sigma u, g)$ for $u \in X^\sigma_B$. Putting $f = (B^*)^\sigma g$ we have $f \in (X^*)^{-\sigma}_{B^*}$ and $F(u) = \langle u, f\rangle_\sigma$. Thus we have shown that $(X^\sigma_B)^* = (X^*)^{-\sigma}_{B^*}$ isometrically via duality pairing (4.11).

Let now $u \in X^\sigma_B$ be given and put $\nu = B^\sigma u \in X$. Then, the Hahn-Banach theorem ensures the existence of some $g \in X^*$ such that $|(\nu, g)| = \|\nu\|\|g\|_*$. Putting $f = (B^*)^{-\sigma} g$ we have $f \in (X^*)^{-\sigma}_{B^*}$ and

$$|\langle u, f\rangle_\sigma| = |(\nu, g)| = \|u\|_\sigma \|f\|_{*,-\sigma}. \tag{4.14}$$

(4.12) and (4.14) together imply that $U = \langle u, \cdot\rangle_\sigma \colon (X^*)^{-\sigma}_{B^*} \to \mathbb{C}$ belongs to $[(X^*)^{-\sigma}_{B^*}]^*$ and $\|U\| = \|u\|_\sigma$. This shows that $X^\sigma_B \hookrightarrow [(X^*)^{-\sigma}_{B^*}]^*$ isometrically.

Let now X be reflexive. For given $U \in [(X^*)^{-\sigma}_{B^*}]^*$, since $(X^*)^{-\sigma}_{B^*}$ and X^* are isometrically to each other, there exists a unique $\nu \in X^{**} = X$ such that

$$U(f) = (\nu, (B^*)^{-\sigma} f) = (B^\sigma u, (B^*)^{-\sigma} f) = \langle u, f\rangle_\sigma$$

where $u = B^{-\sigma}\nu \in X^\sigma_B$. Thus, if X is reflexive, then $X^\sigma_B = [(X^*)^{-\sigma}_{B^*}]^*$ isometrically via the duality pairing (4.11). □

Lemma 4.2.9 (Lemma I.2.9 in [19]).

1. *Given* $\sigma \geq 0$. *Then*

$$\langle u, f\rangle_\sigma = (u, f) \quad \textit{for } u \in X_B^\sigma \textit{ and } f \in X^*.$$

2. *Given* $0 < \sigma < \tau < \infty$. *Then*

$$\langle u, f\rangle_\sigma = \langle u, f\rangle_\tau \quad \textit{for } u \in X_B^\tau \textit{ and } f \in (X^*)_{B^*}^{-\sigma}.$$

Thus the mapping $\langle\cdot,\cdot\rangle\colon \bigcup\limits_{\sigma>0} X_B^\sigma \times (X^*)_{B^*}^{-\sigma} \to \mathbb{C}$ *is well defined in natural way.*

Proof.

$$\begin{aligned}\langle u, f\rangle_\sigma &= (B^\sigma u, (B^*)^{-\sigma} f) = (B^\sigma u, (B^{-\sigma})^* f) = \\ &= (B^{-\sigma} B^\sigma u, f) = (u, f).\end{aligned}$$

$$\begin{aligned}\langle u, f\rangle_\tau &= (B^\tau u, (B^*)^{-\tau} f) = (B^{\tau-\sigma} B^\sigma u, (B^*)^{-(\tau-\sigma)}(B^*)^{-\sigma} f) = \\ &= (B^{\tau-\sigma} B^\sigma u, (B^{-(\tau-\sigma)})^*(B^*)^{-\sigma} f) = \\ &= (B^\sigma u, (B^*)^{-\sigma} f) = \langle u, f\rangle_\sigma.\end{aligned}$$

□

Theorem 4.2.10 (Theorem I.2.10 in [19]).

1. *Let* $\sigma \in [0, \infty)$. *Then*

$$(X_B^{\sigma+})^* = (X^*)_{B^*}^{(-\sigma)-}, \quad [(X^*)_{B^*}^{(-\sigma)-}]^* \hookrightarrow X_B^{\sigma+}$$

and if furthermore X is reflexive, equality "=" holds instead of "$\hookrightarrow$*".*

2. *Let* $\sigma \in [0, \infty)$. *Then*

$$(X_B^{\sigma-})^* = (X^*)_{B^*}^{(-\sigma)+}, \quad [(X^*)_{B^*}^{(-\sigma)+}]^* \hookrightarrow X_B^{\sigma-}$$

and if furthermore X is reflexive, equality "=" holds instead of "$\hookrightarrow$*".*

All the equality relations above hold via the duality pairing $\langle\cdot,\cdot\rangle$.

Proof. The conclusion here directly follows from Theorem 4.2.8, Lemma 4.2.9 and Corollary 4.1.12. □

We now turn to the study of extendibility of the operator A in X to spaces $X_B^{-\sigma}$ ($\sigma > 0$) or their inductive limits or projective limits.

Theorem 4.2.11 (Theorem I.2.11 in [19]). *Suppose that the space X is reflexive. Let $A\colon D(A) \subseteq X \to X$ be a densely defined operator and $A^*\colon D(A^*) \subseteq X^* \to X^*$ its dual operator. Then:*

1. *For given $\sigma, \tau > 0$ the operator A extends uniquely to the continuous operator from $X_B^{-\sigma}$ to $X_B^{-\tau}$ iff $(X^*)^{\tau}_{B^*} \subseteq D(A^*)$, $A^*(X^*)^{\tau}_{B^*} \subseteq (X^*)^{\sigma}_{B^*}$ and $A^*|_{(X^*)^{\tau}_{B^*}}$ is continuous from $(X^*)^{\tau}_{B^*}$ to $(X^*)^{\sigma}_{B^*}$.*
2. *For given $\sigma, \tau \in (0, \infty)$ the operator A extends uniquely to the continuous operator from $X_B^{-\sigma+}$ to $X_B^{-\tau+}$ iff $(X^*)^{\tau-}_{B^*} \subseteq D(A^*)$, $A^*(X^*)^{\tau-}_{B^*} \subseteq (X^*)^{\sigma-}_{B^*}$ and $A^*|_{(X^*)^{\tau-}_{B^*}}$ is continuous from $(X^*)^{\tau-}_{B^*}$ to $(X^*)^{\sigma-}_{B^*}$.*
3. *For given $\sigma, \tau \in [0, \infty)$ the operator A extends uniquely to the continuous operator from $X_B^{-\sigma-}$ to $X_B^{-\tau-}$ iff $(X^*)^{\tau+}_{B^*} \subseteq D(A^*)$, $A^*(X^*)^{\tau+}_{B^*} \subseteq (X^*)^{\sigma+}_{B^*}$ and $A^*|_{(X^*)^{\tau+}_{B^*}}$ is continuous from $(X^*)^{\tau+}_{B^*}$ to $(X^*)^{\sigma+}_{B^*}$.*

Proof. (1) ($\Leftarrow$) Set $\overline{A}_{\sigma,\tau} = (A^*|_{(X^*)^{\tau}_{B^*}})^*$. Then, since $[(X^*)^{\tau}_{B^*}]^* = X_B^{-\tau}$ and $[(X^*)^{\sigma}_{B^*}]^* = X_B^{-\sigma}$ by Theorem 4.2.8 applied to B^* and X^* and by the reflexivity to X, the standard theorem on the dual of a continuous operator from a Banach space to another (see e.g. Corollary 10.6 in [31], p.41) implies that $\overline{A}_{\sigma,\tau}$ is a continuous operator from $X_B^{-\sigma}$ to $X_B^{-\tau}$ and $\|\overline{A}_{\sigma,\tau}\| = \|(A^*|_{(X^*)^{\tau}_{B^*}})\|$. Let us show that $\overline{A}_{\sigma,\tau}$ is an extension of A indeed. In the following $\langle\cdot,\cdot\rangle_{\sigma,*}\colon X_B^{-\sigma} \times (X^*)^{\sigma}_{B^*} \to \mathbb{C}$ ($\sigma > 0$) stands for the duality pairing between $(X^*)^{\sigma}_{B^*}$ and $X_B^{-\sigma}$ and, of course, it has similar properties to $\langle\cdot,\cdot\rangle_{\sigma}$ as stated in Lemma 4.2.9 above; $\langle\cdot,\cdot\rangle_{*}$ is understood similarly to $\langle\cdot,\cdot\rangle$. By definition, we have

$$\langle u, A^*f\rangle_{\sigma,*} = \langle \overline{A}_{\sigma,\tau}u, f\rangle_{\tau,*}, \quad u \in X_B^{-\sigma}, f \in (X^*)^{\tau}_{B^*}.$$

If, $u \in D(A) \subseteq X \subseteq X_B^{-\sigma}$ and $f \in (X^*)^{\tau}_{B^*}$ then

$$\langle u, A^*f\rangle_{\sigma,*} = (u, A^*f) = (Au, f) = \langle Au, f\rangle_{\tau,*}.$$

Thus

$$\langle Au, f\rangle_{\tau,*} = \langle \overline{A}_{\sigma,\tau}u, f\rangle_{\tau,*}, \quad f \in (X^*)^{\tau}_{B^*},$$

which implies that $\overline{A}_{\sigma,\tau}u = Au$. The uniqueness follows from the density of $D(A)$ in X and X in $X_B^{-\sigma}$.

($\Rightarrow$) Assume that A extends the continuous operator from $X_B^{-\sigma}$ to $X_B^{-\tau}$, denoted by $\overline{A}_{\sigma,\tau}$. Set $A^*| = (\overline{A}_{\sigma,\tau})^*$. Then, since $(X_B^{-\sigma})^* = (X^*)^{\sigma}_{B^*}$ and $(X_B^{-\tau})^* = (X^*)^{\tau}_{B^*}$ by Theorem 4.2.8 applied to X^* and B^* and by the reflexivity of X, $A^*|$ is a continuous operator from $(X^*)^{\tau}_{B^*}$ to

$(X^*)^{\sigma}_{B^*}$ and $\|A^*|\ \| = \|\overline{A}_{\sigma,\tau}\|$. If we can show that $(X^*)^{\tau}_{B^*} \subseteq D(A^*)$ and $A^*|_{(X^*)^{\tau}_{B^*}} = A^*|$, we have completed the proof. By definition

$$\langle \overline{A}_{\sigma,\tau}u, f\rangle_{\tau,*} = \langle u, A^*|f\rangle_{\sigma,*}, \quad u \in X_B^{-\sigma}, f \in (X^*)^{\tau}_{B^*}.$$

If, $u \in D(A)$, then

$$(Au, f) = \langle \overline{A}_{\sigma,\tau}u, f\rangle_{\tau,*} = \langle u, A^*|f\rangle_{\sigma,*}, \quad f \in (X^*)^{\tau}_{B^*}$$

which implies that for every $f \in (X^*)^{\tau}_{B^*}$ $(A\cdot, f)\colon D(A) \subseteq X \to \mathbb{C}$ is continuous and therefore $f \in D(A^*)$, thus $(X^*)^{\tau}_{B^*} \subseteq D(A^*)$. Furthermore, for $f \in (X^*)^{\tau}_{B^*}$, the above equation can be rewritten as

$$\langle u, A^*f\rangle_{\sigma,*} = \langle u, A^*|f\rangle_{\sigma,*}, \quad u \in D(A).$$

This together with the density of $D(A)$ in X and $X_B^{-\sigma}$ implies that $A^*f = A^*|f$. Thus $A^*|_{(X^*)^{\tau}_{B^*}} = A^*|$.

(2) ($\Leftarrow$) Set $A^*|_{(X^*)^{\tau-}_{B^*}} = S$. Then, by assumption, S is a continuous operator from $(X^*)^{\tau-}_{B^*}$ to $(X^*)^{\sigma-}_{B^*}$. By Theorem 4.2.10 we have $[(X^*)^{\tau-}_{B^*}]^* = X_B^{-\tau+}$ and $[(X^*)^{\sigma-}_{B^*}]^* = X_B^{-\sigma+}$. The dual operator $S^*\colon X_B^{-\sigma+} \to X_B^{-\tau+}$ is well defined via the duality pairing

$$\langle u, Sf\rangle = \langle S^*u, f\rangle, \quad u \in X_B^{-\sigma+}, f \in (X^*)^{\tau-}_{B^*}. \tag{4.15}$$

A proof similar to the corresponding part of (1) above shows that $S^*|_{D(A)} = A$. Let us prove the continuity of S^*. Given $\sigma' \in (0,\sigma)$. By the continuity of S, Theorem 4.1.10 (b) implies the existence of some $\tau' \in (0,\tau)$ such that

$$\|Sf\|_{\sigma',*} \le C_{\sigma',\tau'}\|f\|_{\tau',*}, \quad f \in (X^*)^{\tau-}_{B^*}. \tag{4.16}$$

Let $u \in X_B^{-\sigma'} \subseteq X_B^{-\sigma+}$. Then (4.15) and (4.16) imply $S^*u \in X_B^{-\tau'}$ and

$$\begin{aligned}\|S^*u\|_{-\tau'} &= \sup_{\substack{f\in(X^*)^{\tau-}_{B^*}\\ \|f\|_{\tau',*}=1}} |\langle S^*u, f\rangle_{\tau',*}| \overset{(\text{since } (X^*)^{\tau-}_{B^*} \text{ is dense in } (X^*)^{\tau'}_{B^*})}{=} \\ &= \sup_{\substack{f\in(X^*)^{\tau-}_{B^*}\\ \|f\|_{\tau',*}=1}} |\langle S^*u, f\rangle| = \sup_{\substack{f\in(X^*)^{\tau-}_{B^*}\\ \|f\|_{\tau',*}=1}} |< u, Sf >| \le \\ &\le C_{\sigma',\tau'}\|u\|_{-\sigma'}\end{aligned}$$

Thus Theorem 4.1.10 (1) is necessary to ensure the continuity of S^*. The uniqueness of the extension follows from the denseness of $D(A)$ in X and X in $X_B^{-\sigma+}$.

($\Rightarrow$) Assume, conversely, that A extends to a continuous operator from $X_B^{-\sigma+}$ to $X_B^{-\tau+}$, denoted T. Define this dual T^* via the duality pairing

$$\langle Tu, f\rangle_* = \langle u, T^* f\rangle, \quad u \in X_B^{-\sigma+}, f \in (X^*)_{B^*}^{\tau-}. \tag{4.17}$$

It is a well-defined operator from $(X^*)_{B^*}^{\tau-}$ to $(X^*)_{B^*}^{\sigma-}$. If $u \in D(A)$ and $f \in (X^*)_{B^*}^{\tau-}$, then

$$(Au, f) = \langle Au, f\rangle_* = \langle Tu, f\rangle_* = \langle u, T^* f\rangle_* = (u, T^* f),$$

which implies that $f \in D(A^*)$ and $A^* f = T^* f$. This proves that $(X^*)_{B^*}^{\tau-} \subseteq D(A^*)$ and $A^*|_{(X^*)_{B^*}^{\tau-}} = T^*$.
Let us show that T^* is continuous. Fix $\sigma' \in (0, \sigma)$. By virtue of the continuity of $T\colon X_B^{-\sigma+} \to X_B^{-\tau+}$, Theorem 4.1.10 (1) guarantees the existence of some $\tau' \in (0, \tau)$ such that

$$\|Tu\|_{-\tau'} \le C_{\tau',\sigma'} \|u\|_{-\sigma'}, \quad u \in X_B^{-\sigma'}. \tag{4.18}$$

For $f \in (X^*)_{B^*}^{\tau'}$, (4.17), (4.18) imply that $T^* f \in (X^*)_{B^*}^{\sigma'}$ and

$$\|T^* f\|_{\sigma',*} = \sup_{\substack{u \in (X)_B^{\sigma-} \\ \|u\|_{-\sigma'} = 1}} |\langle u, T^* f\rangle_{\sigma',*}| = \sup_{\substack{u \in (X)_B^{\sigma-} \\ \|u\|_{-\sigma'} = 1}} |\langle Tu, f\rangle_{\tau',*}| \le$$
$$\le C_{\tau',\sigma'} \|f\|_{\tau',*}.$$

By Theorem 4.1.10 (2) we have the continuity of T^*.

(3) The proof is completely parallel to that for (2) above and is omitted. □

We have a few remarks to the latest theorem:

(a) The conclusion in (2) is still true for $\sigma = \infty$ or $\tau = \infty$ if we assume that $(X^*)_{B^*}^{\tau}$ is dense in X^* in case $\tau = \infty$ and if we replace the concept of continuity of operators by a formally stronger one, as is described in Theorem 4.1.10 (i).
(b) Similarly to (2) and (3) above we also have characterization of operators A in X which is extendibly continuous to one from $X_B^{-\sigma+}$ to $X_B^{-\tau-}$ or from $X_B^{-\sigma-}$ to $X_B^{-\tau+}$ in terms of its dual operator A^*.
(c) Instead of one space X and one operator B we have entirely similar results on the continuous extendibility of an operator $A\colon D(A) \subseteq X \to Y$ to one from $X_B^{-\sigma}$ to $Y_C^{-\tau}$, et al. for two spaces X and Y and operators B and C.

To conclude the present section we put all the results above in a perspective. Take a reflexive Banach space X and an operator B of type (ω, M) therein such that $0 \in \varrho(B)$. Along with the space X and the operator B we have the dual space X^* and the dual operator B^* which have similar properties. Using the domains of the fractional powers B^σ and $(B^*)^\sigma$ we construct the scales of Banach spaces X_B^σ $(\sigma \in \mathbb{R})$ and $(X^*)_{B^*}^\sigma$ and we form the scales of spaces of their inductive limits and projective limits, namely, $X_B^{\sigma+}$ and $(X^*)_{B^*}^{\sigma+}$ $(\sigma \in [-\infty, +\infty))$, $X_B^{\sigma-}$ and $(X^*)_{B^*}^{\sigma-}$ $(\sigma \in (-\infty, +\infty])$. Thus we have the following diagram $(\sigma > 0)$:

$$
\begin{aligned}
&X_B^{-\infty} \supseteq X_B^{-\sigma-} \supseteq X_B^{-\sigma} \supseteq X_B^{-\sigma+} \supseteq X_B^{0-} \supseteq X,\\
&(X^*)_{B^*}^{-\infty} \supseteq (X^*)_{B^*}^{-\sigma-} \supseteq (X^*)_{B^*}^{-\sigma} \supseteq (X^*)_{B^*}^{-\sigma+} \supseteq (X^*)_{B^*}^{0-} \supseteq X^*;\\
&X \supseteq X_B^{0+} \supseteq X_B^{\sigma-} \supseteq X_B^{\sigma} \supseteq X_B^{\sigma+} \supseteq X_B^{\infty},\\
&X^* \supseteq (X^*)_{B^*}^{0+} \supseteq (X^*)_{B^*}^{\sigma-} \supseteq (X^*)_{B^*}^{\sigma} \supseteq (X^*)_{B^*}^{\sigma+} \supseteq (X^*)_{B^*}^{\infty}.
\end{aligned}
$$

If X and X^* are suitable spaces of functions and the operators B and B^* are appropriately taken (usually differential operators), then various classical functions spaces appear as spaces X_B^σ or $(X^*)_{B^*}^\sigma$ $(\sigma > 0)$, and different test function spaces and their corresponding generalized function spaces emerge as the spaces of inductive limits or projective limits with nonnegative indices and nonpositive indices respectively. Thus, we call the spaces to the right of X and (X^*) in the diagram *regular spaces* and those to the left *hyper-spaces*. Theorems 4.2.6 and 4.2.7 clarify the topological structures of all the spaces of inductive limit and projective limit. Theorems 4.2.8 and 4.2.10 establish the duality between the two scales of spaces in the diagram above (i.e. between spaces of smooth functions and generalized functions). Theorem 4.2.11 gives additionally criteria which ensure that an operator initially acting on smooth functions could be extended to spaces of generalized functions. In short, our frame is a kind of Gel'fand-Shilov triple in a Banach space setting. We notice that in general the space to the right of X, in the above diagrams, cannot be embedded into its dual spaces, i.e., the ones to the left of X^*. However, if X is a Hilbert space, this can always be done as long as we identify the duality of the Hilbert space with itself. Also, in some instances, spaces that are „enough" on the right of the X can be embedded to the spaces „enough" on the left of X^*. Anyway, the spaces right to X and X^* together are included in the total of the spaces left to X and X^* together.

An important role in the applications play the inclusions between different Sobolev spaces.

Theorem 4.2.12. *If $\Omega \subseteq \mathbb{R}^n$ is a nonempty bounded open set with the boundary of class C^m with $m \in \mathbb{N}$ and if $0 \leq k < m$, $1 < p \leq q < \infty$ are such that*

$$m - \frac{n}{p} \geq k - \frac{n}{q},$$

then $W^{m,p}(\Omega)$ is continuously injected in $W^{k,q}(\Omega)$.

Theorem 4.2.13 (The Nirenberg-Gagliardo inequality). *If $\Omega \subseteq \mathbb{R}^n$ is a nonempty bounded open set with the boundary of class C^m with $m \in \mathbb{N}$ and if $0 \leq r < k$, $1 < p, q, r < \infty$ and $\theta \in [0,1]$ are such that*

$$k - \frac{n}{q} < (1-\theta)\left(l - \frac{n}{r}\right) + \theta\left(m - \frac{n}{p}\right)$$

and

$$\frac{1}{q} \leq \frac{(1-\theta)}{r} + \frac{\theta}{p},$$

then the inequality

$$\|u\|_{W^{k,q}(\Omega)} \leq C\|u\|^{\theta}_{W^{m,p}(\Omega)}\|u\|^{1-\theta}_{W^{l,r}(\Omega)}, \quad u \in W^{m,p}(\Omega) \cap W^{l,r}(\Omega),$$

holds with a constant $C > 0$ independent of u.

The proof of these theorems can be found in D. Henry [20], p.37 and Proposition 1.2.2, Remark 1.2.1 in J.W. Cholewa, T. Dłotko [9]. The following corollary is consequence of the two previous theorems.

Corollary 4.2.14. *If $\Omega \subseteq \mathbb{R}^n$ is a nonempty bounded open set with the boundary of class C^m with $m \in \mathbb{N}$ and if $0 \leq k < m$, $1 < p \leq q < \infty$ and $\theta \in (\frac{k}{m}, 1]$ are such that*

$$m\theta - \frac{n}{p} < k - \frac{n}{q},$$

then $W^{m,p}(\Omega)$ is continuously injected in $W^{k,q}(\Omega)$ and the inequality

$$\|u\|_{W^{k,q}(\Omega)} \leq C\|u\|^{\theta}_{W^{m,p}(\Omega)}\|u\|^{1-\theta}_{L^{q}(\Omega)}, \quad u \in W^{m,p}(\Omega).$$

holds with a constant $C > 0$ independent of u.

Below we prove a similar result for domains of fractional powers of a positive operator of type M. This and others examples of such embeddings theorems can be found in section 1.6 of D. Henry's book [20].

Theorem 4.2.15 (Theorem 1.6.1 in [20]). *Let Ω be a nonempty bounded open subset of $\mathbb{R}^n$ with the boundary $\partial\Omega$ of class C^m with $m \in \mathbb{N}$. Assume that $p \in (0,\infty)$ and A is a positive operator of type M in $X = L^p(\Omega)$ such that $D(A) = X^1$ is continuously injected in $W^{m,p}(\Omega)$. Then for $\alpha \in (0,1]$ it holds*

$$X^\alpha \subseteq W^{k,q}(\Omega) \quad \textit{if} \quad m\alpha - \frac{n}{p} > k - \frac{n}{q},$$

for $k \in \mathbb{N} \cup \{0\}, p \le q < \infty$, and the natural injection is continuous.

Proof. Let θ be chosen so that

$$m\alpha - \frac{n}{p} > m\theta - \frac{n}{p} > k - \frac{n}{q}.$$

Thus

$$\frac{k}{m} < \theta < \alpha \le 1. \tag{4.19}$$

By the Nirenberg-Gagliardo inequality (see Corollary 4.2.14), we get $W^{m,p}(\Omega) \subseteq W^{k,q}(\Omega)$ and

$$\|u\|_{W^{k,q}(\Omega)} \le C\|u\|^\theta_{W^{m,p}(\Omega)}\|u\|^{1-\theta}_{L^p(\Omega)}, \quad u \in W^{m,p}(\Omega).$$

Since $X^1 = D(A)$ is continuously injected in $W^{m,p}(\Omega)$, we have

$$\|u\|_{W^{k,q}(\Omega)} \le C\|u\|^\theta_1\|u\|^{1-\theta}_{L^p(\Omega)}, \quad u \in X^1. \tag{4.20}$$

Let us consider the identity operator $I\colon L^p(\Omega) \supseteq W^{m,p}(\Omega) \to W^{k,q}(\Omega)$. By (4.19), (4.20) and Exercise 11 in D. Henry's book [20], there exists a constant $K > 0$ such that

$$\|u\|_{W^{k,q}(\Omega)} = \|Iu\|_{W^{k,q}(\Omega)} \le K\|u\|_\alpha, \quad u \in X^{1+\alpha}. \tag{4.21}$$

Fix $u \in X^\alpha$. Since $X^{1+\alpha}$ is a dense subset of X^α, we take $u_n \in X^{1+\alpha}$, $n \in \mathbb{N}$, such that $u_n \to u$ in X^α. Observe that (4.21) implies that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{k,q}(\Omega)$ and thus convergent to some $v \in W^{k,q}(\Omega)$. Since $W^{k,q}(\Omega)$ is continuously injected in $L^p(\Omega)$, we know that $u_n \to v$ in $L^p(\Omega)$. However, for $n \in \mathbb{N}$, we have

$$\|u - v\|_{L^p(\Omega)} \le c\|u - u_n\|_\alpha + \|u_n - v\|_{L^p(\Omega)},$$

so $u = v \in W^{k,q}(\Omega)$. Moreover, we obtain

$$\|u\|_{W^{k,q}(\Omega)} \le K\|u\|_\alpha, \quad u \in X^\alpha,$$

which completes the proof. □

Chapter 5

Examples of scales of Banach spaces

In this section we provide some examples of the regular and hyper-spaces.

Example 5.0.1. Let $X = l^p$ $(1 \le p \le \infty)$. In X we introduce a natural norm

$$\|u\| = \left(\sum_{k=0}^{\infty} |u_k|^p \right)^{\frac{1}{p}}.$$

Consider a sequence of complex numbers $(\lambda_k)_{k \in \mathbb{N}_0}$ such that

$$|\arg \lambda_k| \le \omega, \quad \text{for some } \omega \in [0, \pi) \tag{5.1}$$

and

$$|\lambda_k| \ge a, \quad \text{for some constant } a > 0 \tag{5.2}$$

for all $k \in \mathbb{N}_0$. Having sequence of numbers we define an operator B as follows: $D(B) = \{u = (u_k)_{k \in \mathbb{N}_0} \colon (\lambda_k u_k)_{k \in \mathbb{N}_0} \in l^p\}$ and

$$Bu = (\lambda_k u_k)_{k \in \mathbb{N}_0}.$$

It is easy to see that for all $k \in \mathbb{N}_0$ and $\lambda < 0$

$$|\lambda - \lambda_k| \ge \begin{cases} |\lambda| & \text{if } 0 \le \omega \le \frac{\pi}{2}, \\ |\lambda| \sin \omega & \text{if } \frac{\pi}{2} < \omega < \pi, \end{cases}$$

and

$$|\lambda - \lambda_k| \ge \lambda \sin \omega \text{ for } \lambda \in \Sigma_{\omega+k}.$$

Therefore operator B is of type $(\omega, 1)$ if $0 \le \omega \le \frac{\pi}{2}$ and of type $(\omega, \frac{1}{\sin \omega})$ if $\frac{\pi}{2} < \omega < \pi$. In particular it is m-accretive if $\omega = \frac{\pi}{2}$. Furthermore,

$$(\lambda - B)^{-1} u = ((\lambda - \lambda_k)^{-1} u_k)_{k \in \mathbb{N}}, \quad \text{for } \lambda \in \rho(B).$$

Condition (5.2) implies that $\{\lambda \in \mathbb{C}\colon |\lambda| < a\} \subseteq \rho(B)$. Thus, for $\sigma > 0$, by definition

$$B^{-\sigma}u = \frac{1}{2\pi i}\int_\Gamma \lambda^{-\sigma}(\lambda - B)^{-1}u\, d\lambda = \frac{1}{2\pi i}\int_\Gamma \lambda^{-\sigma}((\lambda-\lambda_k)^{-1}u_k)\, d\lambda =$$
$$= \left(\frac{1}{2\pi i}\int_\Gamma \lambda^{-\sigma}(\lambda-\lambda_k)^{-1}\, d\lambda\cdot u_k\right) = (\lambda^{-\sigma}u_k),$$

where the integration path Γ can be taken $\{\lambda \in \mathbb{C}\colon \arg\lambda - \frac{a}{2} = \Phi\} \subseteq \rho(B)$ for appropriate $\Phi \in (\omega, \pi)$. Therefore, by definition, for $\sigma > 0$

$$X_B^\sigma = l^{p,\sigma}\{\lambda_k\} = \{u = (u_k) \in l^p\colon \|u\|_{p,\sigma} = \left(\sum_{k=0}^\infty (|\lambda_k|^\sigma|u_k|)^p\right)^{\frac{1}{p}} < \infty\}.$$

For $-\sigma < 0$ we have

$$l^{p,-\sigma}\{\lambda_k\} = \{u = (u_k)\colon \|u\|_{p,-\sigma} = \left(\sum_{k=0}^\infty (|\lambda_k|^{-\sigma}|u_k|)^p\right)^{\frac{1}{p}} < \infty\}.$$

Then it is readily seen that $l^{p,-\sigma}\{\lambda_k\}$ is a normed space isometric to l^p under the mapping $l^p \ni (u_k) \mapsto (\lambda_k^\sigma u_k) \in l^{p,-\sigma}\{\lambda_k\}$. So $l^{p,-\sigma}\{\lambda_k\}$ is a Banach space. Moreover, l^p is dense in $l^{p,-\sigma}\{\lambda_k\}$. Namely $X_B^{-\sigma} = l^{p,-\sigma}\{\lambda_k\}$. We set $X_B^0 = l^p = l^{p,0}\{\lambda_k\}$. If no confusion is incurred, $l^{p,\sigma}\{\lambda_k\}$ ($\sigma \in \mathbb{R}$) is abbreviated to $l^{p,\sigma}$. This applies, of course, also to the inductive limits $X_B^{\sigma+} = l^{p,\sigma+}\{\lambda_k\} = \bigcup_{\tau>\sigma} l^{p,\tau}\{\lambda_k\}$ ($\sigma \in [-\infty,\infty)$) and the projective limits $X_B^{\sigma-} = l^{p,\sigma-}\{\lambda_k\} = \bigcap_{\tau<\sigma} l^{p,\tau}\{\lambda_k\}$ ($\sigma \in (-\infty,\infty]$).

For $1 \le p < \infty$, $X^* = (l^p)^* = l^q$ where $q = \frac{p}{p-1}$ and $B^*\colon D(B^*) \subseteq l^q \to l^p$ is defined by

$$D(B^*) = \{u = (u_k) \in l^q\colon (\bar{\lambda}_k u_k) \in l^q\},$$
$$B^*u = (\bar{\lambda}_k u_k).$$

Thus, in accordance to the notations above, $(X^*)_B^\sigma = l^{q,\sigma}\{\bar{\lambda}_k\}$ ($\sigma \in \mathbb{R}$). Then Theorems 4.2.8, 4.2.10 imply that $(l^{p,\sigma})^* = l^{q,-\sigma}$ for all $1 \le p < \infty$ and $\sigma > 0$; $(l^{q,-\sigma})^* = l^{p,\sigma}$ for all $1 < p < \infty$, $\sigma > 0$; $l^{1,\sigma} \hookrightarrow (l^{\infty,-\sigma})^*$; and similarly for the inductive and projective limits.

Remark 5.0.2. Let us consider Cauchy's problem for second order ordinary differential equation

$$\begin{cases} x'' + \lambda x = 0, & t \in (0,\pi),\\ x(0) = x(\pi) = 0. \end{cases}$$

This problem, for $\lambda = \lambda_n = n^2$, has solutions $x_n(t) = \sin nt$.

It shows that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ of eigenvalues for operator $-\frac{\partial^2}{\partial t^2}$ generates the scale of the Banach spaces $l^{p,\sigma}\{\lambda_n\}$.

Example 5.0.3. Let $X = L^p(\mathbb{R}^n)$ with standard norm and define the operator B as follows:

$$\begin{aligned} D(B) &= \{u \in L^p \colon \Lambda(x)u(x) \in L^p\}, \\ Bu &= \Lambda(x)u(x), \end{aligned}$$

where Λ denotes complex-valued measurable function on $\mathbb{R}^n$ satisfying the conditions

$$\begin{aligned} &\Lambda(x) \in L^p_{loc} \\ &|\Lambda(x)| \geq a > 0, \quad \text{for a.e. } x \in \mathbb{R}^n, \\ &|\arg \Lambda(x)| \leq \omega, \quad \text{for a.e. } x \in \mathbb{R}^n, \text{ with } \omega \in [0, \pi). \end{aligned}$$

Then, using the same arguments as in the previous Example, we can prove that:

$$\begin{aligned} (B^{-\sigma})(x) &= [\Lambda(x)]^{-\sigma}u(x), \quad \text{for } \sigma > 0, \\ X_B^{\sigma} &= L^{p,\sigma}\{\Lambda(x)\} = \{u \in L^p \colon \left(\int (|\lambda(x)|^{\sigma}|u(x)|)^p\, dx\right)^{\frac{1}{p}} < \infty\}, \\ X_B^0 &= L^p, \\ X_B^{-\sigma} &= L^{p,-\sigma}\{\Lambda(x)\} = \\ &= \{u\text{-measurable on } \mathbb{R}^n \colon \left(\int (|\lambda(x)|^{-\sigma}|u(x)|)^p\, dx\right)^{\frac{1}{p}} < \infty\}, \\ \|u\|_{p,\sigma} &= \left(\int (|\lambda(x)|^{\sigma}|u(x)|)^p\, dx\right)^{\frac{1}{p}} \end{aligned}$$

Example 5.0.4. Let $X = L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ with the usual norm $\|\cdot\|_p$. The operator B in L^p is given by

$$\begin{aligned} D(B) &= \{u \in L^p \colon \ \Delta u \in L^p\}, \\ Bu &= (I - \Delta)u. \end{aligned}$$

We note that the operator B is well-defined on $\mathcal{S}$ and $\mathcal{S}'$, the Schwartz test functions space and the tempered distribution space. Let F and F^{-1} be the

Fourier transform and its inverse, which acts on $\mathcal{S}$ and $\mathcal{S}'$ continuously. We have

$$F(\lambda I - B)u = (\lambda - 1 - x^2)Fu, \quad u \in \mathcal{S}'.$$

Therefore, if $\lambda \in [1, \infty)$ then $(\lambda I - B)\colon \mathcal{S}' \to \mathcal{S}'$ is invertible and

$$(\lambda I - B)^{-1}u = F^{-1}[(\lambda - 1 - x^2)Fu] = \frac{1}{(2\pi)^{n/2}} F^{-1}[(\lambda - 1 - x^2)] * u.$$

According to Young's inequality we have

$$\|(\lambda I - B)^{-1}u\|_p \leq (-\mathrm{Re}\lambda + 1)^{-1}\|u\|_p, \quad u \in L^p.$$

For $\sigma > 0$ and $u \in L^p$, by definition

$$\begin{aligned}(B^{-\sigma}u)(x) &= \frac{1}{2\pi i}\int_\Gamma \lambda^{-\sigma}[(\lambda I - B)^{-1}u](x)\, d\lambda = \\ &= \frac{1}{2\pi i}\int_\Gamma \lambda^{-\sigma} F^{-1}[(\lambda - 1 - x^2)Fu]\, d\lambda = \\ &= F^{-1}\left[\frac{1}{2\pi i}\int_\Gamma \lambda^{-\sigma}(\lambda - 1 - x^2)\, d\lambda Fu\right] = \\ &= F^{-1}[(1 + x^2)^{-\sigma}Fu].\end{aligned}$$

Therefore,

$$X_B^{\sigma/2} = W^{\sigma,p} = \{u \in L^p\colon \|F^{-1}(1 + x^2)^{\sigma/2}Fu\|_p < \infty\}.$$

For $-\sigma < 0$ we have

$$W^{-\sigma,p} = \{u\colon F^{-1}(1 + x^2)^{-\sigma/2}Fu \in L^p\}$$

with the norm

$$\|u\|_{-\sigma,p} = \|F^{-1}(1 + x^2)^{-\sigma/2}Fu\|_p.$$

Since each $W^{-\sigma,p}$ is isometric to L^p under the mapping

$$F^{-1}(1 + x^2)^{-\sigma/2}F\colon W^{-\sigma,p} \to L^p,$$

it is a Banach space itself. We can show the density of L^p in $W^{-\sigma,p}$. So $X_B^{-\sigma/2} = W^{-\sigma,p}$.

Next we formulate two easy propositions. The proofs can be found in Liu Gui-Zhong [19].

Proposition 5.0.5. *If* $F^{-1}(1 + x^2)^{(\tau-\sigma)/2} \in L^{q/2}$, $q = \frac{p}{p-1}$, *then* $W^{\sigma,p} \hookrightarrow W^{\tau,q}$.

Proposition 5.0.6. *If $p \leq \frac{4}{3}$, $q \geq 4$ and $\frac{(\sigma-\tau)p}{n(2-p)} > 1$, then $W^{\sigma,p} \hookrightarrow W^{\tau,q}$.*

We can find similar theorem in R.A. Adams [1] (Theorem 7.63 (c), (d), p.221)

Theorem 5.0.7.

(i) *If $\tau < \sigma$ then $W^{\sigma,p} \subseteq W^{\tau,p}$,*

(ii) *$\tau \leq \sigma$ and if either $1 < p \leq q \leq \frac{np}{n-(\sigma-\tau)p} < \infty$ or $p = 1$ and $1 < q \leq \frac{n}{n-\sigma+\tau} < \infty$ then $W^{\sigma,p} \subseteq W^{\tau,q}$.*

The spaces $W^{\sigma,p}$ are called *fractional Sobolev spaces.* The next theorem informs us that when σ is an integer, then the spaces $W^{\sigma,p}$ coincide with the classical Sobolev spaces. This property was first proved by A.P. Calderón.

Theorem 5.0.8. *For $p \in (1,\infty)$ and k-positive integer we have*

$$W^{k,p} = \{u \in L^p \colon D^\alpha u \in L^p, |\alpha| \leq k\}$$

with equivalent norm

$$\sum_{|\alpha|\leq k} \|D^\alpha u\|_p.$$

Proof of the theorem above can be found in the chapter V section 3.4 of Stein [35].

Example 5.0.9 (Example VI in [19]). Let $X = L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ with the usual norm $\|u\|_p = \left(\int_{\mathbb{R}} |u|^p \, dx\right)^{\frac{1}{p}}$. Consider the Heat-Diffusion Equation in the space of tempered distributions $\mathcal{D}'$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{5.3}$$

Here the differentiation with respect to x is understood in the sense of the tempered distributions, while that with respect to t is of the topology $\mathcal{S}'$. Since the Fourier transform F and its inverse are continuous on $\mathcal{D}'$, the above equation (5.3) is equivalent to

$$\frac{\partial(Fu)}{\partial t} = -x^2 Fu.$$

Hence the formula for solution of the initial value problem of (5.3) easily follows

$$e^{t\frac{\partial^2}{\partial x^2}} u = F^{-1} e^{-tx^2} Fu = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} * u.$$

If $u \in L^p$, then $e^{t\frac{\partial^2}{\partial x^2}}u = v(x)$ extends to an entire function $v(\zeta)$ ($\zeta = x+iy$) which is given by

$$v(\zeta) = v(x+iy) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{(x+iy)^2}{4t}} * u(x) = \frac{1}{2\sqrt{\pi t}}e^{\frac{y^2}{4t}}\int_{\mathbb{R}} e^{-\frac{(x-\zeta)^2+2i(x-\zeta)y}{4t}}u(\zeta)\,d\zeta.$$

Furthermore, Young's inequality leads to the estimate

$$\|v(x+iy)\|_{p,x} \le e^{\frac{y^2}{4t}}\|u\|_p.$$

Indeed,

$$\frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}}\left|e^{-\frac{(x-\zeta)^2+2i(x-\zeta)y}{4t}}\right| d\zeta = \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}} e^{-\frac{\zeta^2}{4t}}\,d\zeta = 1.$$

Put $B = \left(e^{\frac{\partial^2}{\partial x^2}}\right)^{-1}$ with $D(B) = R(e^{\frac{\partial^2}{\partial x^2}}|_{L^p})$. It is not difficult to show that the operator B is of the type $(0, M)$ for some $M \ge 0$ and for $t > 0$, $B^t = \left(e^{t\frac{\partial^2}{\partial x^2}}|_{L^p}\right)^{-1}$. Therefore, by definition, $X_B^t = R(e^{t\frac{\partial^2}{\partial x^2}}|_{L^p})$ for $t > 0$. Thus, we need to characterize $R(e^{t\frac{\partial^2}{\partial x^2}}|_{L^p})$ explicitly.

In view of the above consideration, let us assume that $v(\zeta) = v(x+iy)$ is an entire function such that

$$\|v(x+iy)\|_{p,x} \le Me^{sy^2}, \quad x, y \in \mathbb{R}, \tag{5.4}$$

where M and s are nonnegative constants. We intend to find some $t > 0$ and $u \in L^p$ such that $v(x) = e^{t\frac{\partial^2}{\partial x^2}}u(x)$. A heuristic consideration suggests the following candidate for u

$$u(x) = \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}} e^{-\frac{(\eta-ix)^2}{4t}}v(i\eta)\,d\eta = \frac{1}{2\sqrt{\pi t}i}\int_{-\infty i}^{\infty i} e^{-\frac{(x-\zeta)^2}{4t}}v(\zeta)\,d\zeta. \tag{5.5}$$

Of course, we must prove now the convergence of the above integrals for $u \in L^p$ and $e^{t\frac{\partial^2}{\partial x^2}}u = v(x)$.

Lemma 5.0.10. *If $v(\zeta)$ is an entire function satisfying condition* (5.4) *above, then for any $s' > s$ there exists $\alpha = \alpha(s, s', p)$ such that*

$$\sup_{x+iy\in\mathbb{C}} |v(x+iy)| \le \alpha Me^{s'y^2}.$$

Proof. By the mean value theorem we have ($R > 0$)

$$v(x+iy) = \frac{1}{\pi R^2} \int_{|\zeta+i\eta|<R} v\left((x+\zeta)+i(y+\eta)\right) \, d\zeta d\eta.$$

An application of Hölder inequality leads to

$$|v(x+iy)| \leq \frac{1}{\pi R^2} (\pi R^2)^{1-\frac{1}{p}} \left(\int_{|\zeta+i\eta|<R} |v\left((x+\zeta)+i(y+\eta)\right)|^p \, d\zeta d\eta \right)^{\frac{1}{p}}.$$

□

According to the lemma above we are now certain that for each entire function v satisfying (5.4) for some $s > 0$, the integral (5.5) converges as long as $0 < t < \frac{1}{4s}$. Furthermore, the Cauchy integral theorem enables us to transfer the path of integration so that ($c \in \mathbb{R}$)

$$u(x) = \frac{1}{2\sqrt{\pi t} i} \int_{c-\infty}^{c+\infty} e^{\frac{1}{4t}(x-\zeta)^2} v(\zeta) \, d\zeta. \tag{5.6}$$

In particular, for $c = x$ we have

$$u(x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}\eta^2} v(x+i\eta) \, d\eta. \tag{5.7}$$

We omit proofs of these lemmas, because they are rather straightforward. We can find them in [19].

Lemma 5.0.11. *Let v be an entire function satisfying condition* (5.4) *As in the lemma above. Then, for $t < \frac{1}{4s}$, function $u(x)$ in* (5.5) *is well-defined and is given equivalently by* (5.6) *and* (5.7). *Moreover, $u \in L^p$ and for any $a \in (s, \frac{1}{4t})$ the following statements holds*

$$\|u\|_p \leq \frac{M}{2\sqrt{\pi t}} \|e^{-(\frac{1}{4t}-a)\eta^2}\|_q \|e^{-(a-s)\eta^2}\|_p. \tag{5.8}$$

Lemma 5.0.12. *Assume that all the conditions in the above Lemma* 5.0.11 *are satisfied. Then*

$$e^{t\frac{\partial^2}{\partial x^2}} u(x) = v(x).$$

Definition 5.0.13. Let $p \geq 1$ and $s > 0$. Let $A^{p,s}$ denote the normed space of entire functions $v(\zeta)$ such that

$$|v|_{p,s} = \sup_{y\in\mathbb{R}} e^{-sy^2} \left(\int_{\mathbb{R}} |v(x+iy)|^p \, dx \right)^{\frac{1}{p}} \leq \infty. \tag{5.9}$$

Proposition 5.0.14. *For each $p \geq 2$ and $s > 0$, the space $A^{p,s}$ is a Banach space.*

Proof. Let (v_n) be a Cauchy sequence in $A^{p,s}$. Then for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$|v_n - v_m|_{p,s} \leq \sup_{y\in\mathbb{R}} e^{-sy^2} \left(\int_{\mathbb{R}} |v_n(x+iy) - v_m(x+iy)|^p \, dx \right)^{\frac{1}{p}} \leq \varepsilon, \quad (5.10)$$

for $n, m \in \mathbb{N}$. Then, for any given $s' > s$, Lemma 5.0.10 implies that

$$\sup_{x+iy\in\mathbb{C}} |v_n(x+iy) - v_m(x+iy)| \leq \alpha\varepsilon e^{s'y^2}.$$

This estimate shows that the sequence of functions $(v_m(x+iy))$ converges to an entire function $\bar{v}(x+iy)$ uniformly on each strip $\{x+iy\colon |y| \leq b\}$ $(b > 0)$. Upon fixing n and letting $m \to \infty$ in (5.10), in view of Lebesgue's dominance converge theorem we conclude that $\bar{v} \in A^{p,s}$ and $v_n \to \bar{v}$ in $A^{p,s}$. Thus the space $A^{p,s}$ is complete. □

Definition 5.0.15. For $s \in (0, \infty]$, let $A^{p,s+} = \bigcup_{\sigma<s} A^{p,\sigma}$ be the inductive limit of the family of Banach space $\{A^{p,\sigma}\colon \sigma < s\}$.
For $s \in [0, \infty)$, let $A^{p,s-} = \bigcap_{\sigma>s} A^{p,\sigma}$ be the projective limit of the family of Banach space $\{A^{p,\sigma}\colon \sigma > s\}$.

As summary of the above discussion we obtain:

Theorem 5.0.16. *For $t \in [0, \infty)$*

$$(L^p)_B^{t+} = A^{p,(\frac{1}{4t})+} \quad \text{topologically.}$$

For $t \in (0, \infty]$

$$(L^p)_B^{t-} = A^{p,(\frac{1}{4t})-} \quad \text{topologically.}$$

Chapter 6

Sectorial Operators

Let us denote a sector of the complex plane:

$$S_{a,\phi} = \{\lambda \in \mathbb{C}\colon \phi \le |\arg(\lambda - a)| \le \pi, \lambda \ne a\} \tag{6.1}$$

where $a \in \mathbb{R}$ and $\phi \in (0, \frac{\pi}{2})$.

Definition 6.0.1. Consider a linear closed and densely defined operator $A\colon X \supseteq D(A) \to X$ acting in a Banach space X. Then A is a *sectorial operator* in X if and only if there exists $a \in \mathbb{R}$, $\phi \in (0, \frac{\pi}{2})$ and $M > 0$ such that the resolvent set $\rho(A)$ contains the sector $S_{a,\phi}$ and

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X,X)} \le \frac{M}{|\lambda - a|}, \quad \text{for each } \lambda \in S_{a,\phi}. \tag{6.2}$$

We shall remark, that when A is a sectorial operator, $A_\omega = A + \omega$ for an arbitrary $\omega \in \mathbb{R}$ is also a sectorial operator. In addition, we have

$$\mathrm{Re}\sigma(A_\omega) \ge a + \omega,$$

so it is always possible to choose $\omega > 0$ for which $\mathrm{Re}\sigma(A_\omega) > 0$.

Now we introduce some equivalent conditions for being a sectorial operator. For proof, we refer to an example to book of J. Cholewa, T. Dłotko [9], Proposition 1.3.1 (p.33).

Proposition 6.0.2 (Proposition 1.3.1 in [9] and Theorem 2.2.4 in [12]). *Let $A\colon X \supseteq D(A) \to X$ be a linear closed and densely defined operator in a Banach space X and consider operators $A_\omega = A + \omega$ with $\omega \in \mathbb{R}$. Then the following conditions are equivalent:*

(a) *A_ω is sectorial in X for some $\omega \in \mathbb{R}$,*
(b) *A_ω is sectorial in X for each $\omega \in \mathbb{R}$,*

(c) *There exists $M > 1$, $k, \omega \in \mathbb{R}$ such that the resolvent set $\rho(A_\omega)$ of A_ω contains a half plane $\{\lambda \in \mathbb{C}\colon \mathrm{Re}\lambda \le k\}$ and*

$$\|\lambda(\lambda - A_\omega)^{-1}\| \le M, \quad \text{for } \mathrm{Re}\lambda \le k.$$

Remark 6.0.3. Let $\phi \in (0, \frac{\pi}{2})$ and $a \in \mathbb{R}$. Let $A\colon X \supseteq D(A) \to X$ be a linear closed and densely defined operator in a Banach space X. In the previous sections we have introduced the notion of operator of type $(\omega, M(\theta))$. Of course, if the operator A is sectorial with sector $S_{0,\omega}$, then it is of type (ω, M). Now we remark that when A is of the type $(\omega, M(\theta))$, then A is sectorial in X with sector $S_{0,\omega-\varepsilon}$, for some $\varepsilon > 0$.

Proof. Fix arbitrary $\varepsilon \in (0, \pi - \omega)$ and $\theta \in [\omega+\varepsilon, 2\pi-\omega-\varepsilon]$. Take $\alpha \in (0, \frac{\pi}{2})$ such that $\sin\alpha \le \frac{1}{2M(\theta)}$ and set

$$V_\theta = (\theta - \alpha, \theta + \alpha) \cap [\omega + \varepsilon, 2\pi - \omega - \varepsilon], \quad U_\theta = \{\lambda \in \mathbb{C}\colon \arg\lambda \in V_\theta\}.$$

Take $\lambda \in U_\theta \setminus \{0\}$ and let $\lambda_0 \in \mathbb{C}$ with $\arg\lambda_0 = \theta$ be chosen so that λ is the projection of λ_0 on the ray $\{z \in \mathbb{C}\colon \arg z = \arg\lambda\}$. Thus $\lambda_0 \in \rho(A)$ and $|\lambda_0| \ge |\lambda|$. Moreover, we have

$$|\lambda - \lambda_0| \le |\lambda_0|\sin\alpha \le \frac{|\lambda_0|}{2M(\theta)} \le \frac{1}{2\|(\lambda_0 - A)^{-1}\|} \le \frac{1}{\|(\lambda_0 - A)^{-1}\|}.$$

Using Theorem 1.1.11 in Czaja [12] (p.14), we obtain $\lambda \in \rho(A)$ and

$$\begin{aligned}\|(\lambda - A)^{-1}\| &\le \sum_{n=0}^{\infty}(|\lambda - \lambda_0|\|(\lambda_0 - A)^{-1}\|)^n\|(\lambda_0 - A)^{-1}\| \le \\ &\le \sum_{n==0}^{\infty}\left(\frac{1}{2}\right)^n \frac{M(\theta)}{|\lambda_0|} \le \frac{2M(\theta)}{|\lambda|}.\end{aligned}$$

Therefore for all $\theta \in [\omega+\varepsilon, 2\pi-\omega-\varepsilon]$, there exists a V_θ-open neighbourhood of θ in $[\omega + \varepsilon, 2\pi - \omega - \varepsilon]$ and a constant $M(\theta) \ge 1$ such that

$$\|(\lambda - A)^{-1}\| \le \frac{2M(\theta)}{|\lambda|}, \quad \text{for } \lambda \in \mathbb{C} \setminus \{0\}, \arg\lambda \in V_\theta.$$

By the compactness of $[-\pi + \omega, \pi - \omega]$, we see that there exists $M \ge 1$ such that

$$\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda|}, \quad \text{for } \lambda \in \mathbb{C} \setminus \{0\}, |\arg\lambda| \ge \omega + \varepsilon,$$

so A is sectorial in X with the sector $S_{0,\omega+\varepsilon}$. □

6.1 Examples of Sectorial Operators

In this section we want to introduce some examples of sectorial operators. Examples of sectorial operators are analysed in particular in J.W. Cholewa, T. Dłotko [9], D. Henry [20] and R. Czaja [12]. We start with two simple standard examples.

Example 6.1.1. Each bounded linear operator defined on a Banach space X is sectorial.

Proof. If A is bounded linear operator in X then

$$\{\lambda \in \mathbb{C}\colon |\lambda| > \|A\|_{\mathcal{L}(X,X)}\} \subseteq \rho(A) \quad \text{and} \quad (\lambda - A)^{-1} = \sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}}.$$

In particular, the half plane $\{\lambda \in \mathbb{C}\colon \mathrm{Re}\lambda \leq -2\|A\|_{\mathcal{L}(X,X)}\}$ is contained in $\rho(A)$ and

$$\|\lambda(\lambda - A)^{-1}\|_{\mathcal{L}(X,X)} \leq \sum_{n=0}^{\infty} \left(\frac{\|A\|_{\mathcal{L}(X,X)}}{|\lambda|}\right)^n \leq \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2.$$

Therefore A is sectorial as a consequence of Proposition 6.0.2. □

Example 6.1.2. If X, Y are Banach spaces and A is sectorial in X, B is sectorial in Y, then the product operator $(A,B)\colon D(A) \times D(B) \to X \times Y$, where $(A,B)(x,y) = (Ax, By)$, is sectorial in $X \times Y$.

Proof. Let $S_{a,\phi}$ and $S_{b,\psi}$ be the sectors given respectively for A and B. Define $c = \min(a,b)$, $C = (A,B)$, and $C_{-c} = (A - cI, b - CI)$. Then $\{\lambda \in \mathbb{C}\colon \mathrm{Re}\lambda < 0\}$ is a subset of $\rho(C)$ and

$$\|(\lambda I - C)^{-1}\|_{\mathcal{L}(X\times Y, X\times Y)} \leq \frac{const.}{|\lambda|}$$

for $\mathrm{Re}\lambda < 0$. Hence C is sectorial in $X \times Y$ as a result of Proposition 6.0.2. □

Proposition 6.1.3 (Perturbation result, Proposition 1.3.2 in [9]). *Let $A\colon D(A) \to X$ be a sectorial operator in a Banach space X. Let us consider a closed, linear operator $B\colon D(B) \to X$ such that $D(A) \subseteq D(B) \subseteq X$ and let B is subordinated to A according to the condition*

$$\|Bv\|_X \leq c\|Av\|_X + c'\|v\|_X, \quad v \in X. \tag{6.3}$$

If the condition (6.3) holds with $c \leq M_0$ (M_0 is defined in (6.8)), then the perturbated operator $A + B$ with $D(A+B) = D(A)$ is sectorial in X.

Proof. Based on Proposition 6.0.2 take $k < 0$, $\omega \in \mathbb{R}$ such that, for $A_\omega = A + \omega I$,

$$\{\lambda \in \mathbb{C}\colon \mathrm{Re}\lambda \le k\} \subseteq \rho(A_\omega)$$

and

$$\|\lambda(\lambda I - A_\omega)^{-1}\|_{\mathcal{L}(X,X)} \le M, \quad \text{for } \mathrm{Re}\lambda \le k. \tag{6.4}$$

Condition (6.3) then leads to

$$\|Bv\|_X \le c\|A_\omega v\|_X + c''\|v\|_X, \quad v \in X, \tag{6.5}$$

where $c'' = c' + c|\omega|$.

The crucial step of the proof is to show that there exists $k_0 \le k$ such that

$$1 \in \rho\left(B(\lambda I - A_\omega)^{-1}\right), \quad \mathrm{re}\lambda \le k_0. \tag{6.6}$$

For this we shall use (6.4) and (6.5), estimating as follows

$$\begin{aligned}
&\|B(\lambda I - A_\omega)^{-1}v\|_X \le \\
&\quad \le c\|A_\omega(\lambda I - A_\omega)^{-1}v\|_X + c''\|(\lambda I - A_\omega)^{-1}v\|_X \le \\
&\quad \le c\|(\lambda I - A_\omega)(\lambda I - A_\omega)^{-1}v\|_X + \\
&\qquad + c\|\lambda(\lambda I - A_\omega)^{-1}v\|_X + c''\|(\lambda I - A_\omega)^{-1}v\|_X \le \\
&\quad \le \left(c(1+M) + \frac{c''M}{|\lambda|}\right)\|v\|_X, \quad v \in X.
\end{aligned} \tag{6.7}$$

Under the restrictions

$$c \le \frac{1}{2(1+M)} =: M_0, \quad 4c''M \le |\lambda| \tag{6.8}$$

we further have from (6.7) inequality

$$\|B(\lambda I - A_\omega)^{-1}\|_{\mathcal{L}(X,X)} \le \frac{3}{4}. \tag{6.9}$$

Since the spectral radius of a bounded linear operator on X does not exceed its $\mathcal{L}(X,X)$ norm (see K. Yosida [41], chapter VIII, section 2), condition (6.9) shows that number 1 is contained in the resolvent set of $B(\lambda I - A_\omega)^{-1}$, i.e. (6.6) is proved with

$$k_0 = \min(k, -4c''M).$$

Then, for $\mathrm{Re}\lambda \le k_0$, we have:

$$\begin{aligned}
(\lambda I - (A_\omega + B))^{-1} &= \left((I - B(\lambda I - A_\omega)^{-1})(\lambda I - A_\omega)\right)^{-1} = \\
&= (\lambda I - A_\omega)^{-1}(I - B(\lambda I - A_\omega)^{-1})^{-1},
\end{aligned}$$

which, in presence of (6.4) and (6.9), leads to the estimate

$$\begin{aligned}
&\|(\lambda I - (A_\omega + B))^{-1}\|_{\mathcal{L}(X,X)} \le \\
&\le \|(\lambda I - A_\omega)^{-1}\|_{\mathcal{L}(X,X)} \|(I - B(\lambda I - A_\omega)^{-1})^{-1}\|_{\mathcal{L}(X,X)} \le \frac{4M}{|\lambda|}.
\end{aligned}$$

Since $A_\omega + B = (A+B)_\omega$, the operator $A+B$ fulfills the requirements of Proposition 6.0.2, what completes the proof. □

The next result provides us with a lot of examples of sectorial operators.

Proposition 6.1.4 (Proposition 1.3.3 in [9] and Theorem 5.1.2 in [12]). *Let $H\colon H \supseteq D(A) \to H$ be a densely defined, linear, selfadjoint operator in a Hilbert space H. If, in addition, A is bounded below in H, that is there exists $m \in \mathbb{R}$ such that*

$$\langle Ax, x\rangle_H \ge m\|x\|_H^2, \quad x \in D(A), \tag{6.10}$$

then A is a sectorial operator in H.

Proof. Let us recall that the spectrum of a selfadjoint operator is contained in the real axis. Moreover, since A is bounded below, $\sigma(A)$ have to be contained in the interval $[m,\infty)$. This implies in particular that the sector

$$S_{m,\frac{\pi}{4}} = \left\{\lambda \in \mathbb{C}\colon \frac{\pi}{4} \le |\arg(\lambda - m)| \le \pi, \lambda \ne m\right\}$$

is contained in the resolvent set of A.

We will now prove, for $\lambda \in S_{n,\frac{\pi}{P4}}$, the validity of the estimate (6.2). Let $\lambda \in S_{m,\frac{\pi}{4}}$ and take $\lambda' = \lambda - m$. Only the following cases are the only possibilities.

Case 1: $\lambda = \lambda' + m$, where $\mathrm{Re}\lambda' < 0$. In this case, since $A - mI$ must be symmetric and nonnegative whereas $-2\mathrm{Re}\lambda' > 0$, we obtain:

$$\begin{aligned}
\|(\lambda I - A)x\|_H^2 &= \|(\lambda' I - (A - mI))x\|_H^2 = \\
&= |\lambda'|^2\|x\|_H^2 - 2\mathrm{Re}\lambda'\langle (A - nI)x, x\rangle_H + \|(A - nI)x\|_H^2 \ge \\
&\ge |\lambda'|^2\|x\|_H^2.
\end{aligned}$$

Case 2: $\lambda = \lambda' + m$, where $|\mathrm{Im}\lambda'| \ge |\mathrm{Re}\lambda'|$. Then we have:

$$\begin{aligned}
\|(\lambda I - A)x\|_H^2 &= \|(\lambda' I - (A - mI))x\|_H^2 = \\
&= |\mathrm{Im}\lambda'|^2\|x\|_H^2 + \|(\mathrm{Re}\lambda' I - (A - mI))x\|_H^2 \ge \\
&\ge |\mathrm{Im}\lambda'|^2\|x\|_H^2 \ge \frac{|\lambda'|^2}{2}\|x\|_H^2.
\end{aligned}$$

As a result of the two inequalities above it is seen that

$$\|(\lambda I - A)x\|_H \geq \frac{|\lambda - m|}{\sqrt{2}} \|x\|_H,$$

for each $\lambda \in S_{m,\frac{\pi}{2}}$, $x \in D(A)$, which is the counterpart of (6.2). □

The most important among applications is the case when the constant m in condition (6.10) is positive. Such operators we called *positive definite.*

Now, basing on Proposition 6.1.4, we introduce some important examples of positive, sectorial operators.

Example 6.1.5. The unbounded operator $I - \Delta\colon H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is positive definite and sectorial.

Proof. For the functions $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$ let us choose an open ball $B(0,r)$ in $\mathbb{R}^n$ with centre in 0 and radius r, which contains their supports. Integrating twice by parts we obtain:

$$\begin{aligned}\int_{\mathbb{R}^n} [(I-\Delta)\phi(x)]\overline{\psi(x)}\, dx &= \int_{\mathbb{R}^n} \phi(x)\overline{\psi(x)}\, dx - \int_{B(0,r)} \Delta\phi(x)\overline{\psi(x)}\, dx = \\ &= \int_{\mathbb{R}^n} \phi(x)\overline{\psi(x)}\, dx - \int_{B(0,r)} \phi(x)\overline{\Delta\psi(x)}\, dx = \\ &= \int_{\mathbb{R}^n} \phi(x)\overline{[(I-\Delta)\psi(x)]}\, dx,\end{aligned}$$

so

$$((I-\Delta)\phi, \psi)_{L^2(\mathbb{R}^n)} = (\phi, (I-\Delta)\psi)_{L^2(\mathbb{R}^n)}, \quad \phi, \psi \in C_0^\infty(\mathbb{R}^n). \tag{6.11}$$

Moreover, integration by parts gives us:

$$\begin{aligned}\int_{\mathbb{R}^n} [(I-\Delta)\phi(x)]\overline{\phi(x)}\, dx &= \int_{\mathbb{R}^n} |\phi(x)|^2\, dx - \int_{B(0,r)} \Delta\phi(x)\overline{\phi(x)}\, dx = \\ &= \int_{\mathbb{R}^n} |\phi(x)|^2\, dx + \int_{B(0,r)} |\nabla\phi(x)|^2\, dx \geq \\ &\geq \int_{\mathbb{R}^n} |\phi(x)|^2\, dx,\end{aligned}$$

where $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$, and thus

$$((I-\Delta)\phi, \phi)_{L^2(\mathbb{R}^n)} \geq \|\phi\|^2_{L^2(\mathbb{R}^n)}, \quad \phi \in C_0^\infty(\mathbb{R}^n). \tag{6.12}$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $H^2(\mathbb{R}^n)$, (6.11) and (6.12) ensure that $I - \Delta$ on $H^2(\mathbb{R}^n)$ is a positive definite symmetric operator in $L^2(\mathbb{R}^n)$. Let us consider Fourier transform F and let us recall that it is an isomorphism of the space $\mathcal{S} \subseteq H^2(\mathbb{R}^n)$ of rapidly decreasing complex-valued functions. We denote its inverse on $\mathcal{S}$ by F^{-1}. Take $\phi \in C_0^\infty \subseteq \mathcal{S}$ and set

$$h(x) = F^{-1}\left(\frac{1}{1+|\xi|^2}F\phi(\xi)\right)(x), \quad x \in \mathbb{R}^n.$$

Note, that $h \in \mathcal{S}$ and

$$F[(I-\Delta)h](\xi) = Fh(\xi) - F\Delta h(\xi) = \frac{1}{1+|\xi|^2}F\phi(\xi) + |\xi|^2 Fh(\xi) = F\phi(\xi),$$

for $\xi \in \mathbb{R}^n$.

Applying F^{-1} we obtain $[I - \Delta]h = \phi$. This shows that $R(I - \Delta)$ is dense in $L^2(\mathbb{R}^n)$, since it contains $C_0^\infty(\mathbb{R}^n)$. In addition, we infer from the density of $C_0^\infty(\mathbb{R}^n)$ in $H^2(\mathbb{R}^n)$, (6.12) and Schwarz inequality that

$$\|(I-\Delta)\phi\|_{L^2(\mathbb{R}^n)} \geq \|\phi\|_{L^2(\mathbb{R}^n)}, \quad \phi \in H^2(\mathbb{R}^n).$$

From this we conclude that $(I - \Delta)\phi = 0$ implies $\phi = 0$, and thus $(I - \Delta)^{-1}\colon R(I - \Delta) \to H^2(\mathbb{R}^n)$ exists and is bounded in $L^2(\mathbb{R}^n)$. Hence $1 \in \rho(\Delta)$. Since the Laplace operator considered in $L^2(\mathbb{R}^n)$ on the domain $H^2(\mathbb{R}^n)$ is closed, the resolvent operator is closed and bounded, that is its domain $R(I - \Delta)$ is closed in $L^2(\mathbb{R}^n)$. This shows that $R(I - \Delta) = L^2(\mathbb{R}^n)$ and as a consequence of Proposition 6.1.4 and the fact that the linear operator in Hilbert space H is self-adjoint whenever it is symmetric and its range coincides with H, $I - \Delta$ is sectorial and positive definite operator. □

We omit the proofs of next two examples, which we can find for example in the paper of J. Cholewa, T. Dłotko [9] and R. Czaja [12].

Example 6.1.6. $-\Delta\colon H^2(\Omega)\cap H_0^1(\Omega) \to L^2(\Omega)$, where $\partial\Omega \in C^2$, is sectorial and positive definite.

Example 6.1.7. $\Delta^2\colon H^4(\Omega)\cap H_0^2(\Omega) \to L^2(\Omega)$, where $\partial\Omega \in C^4$, is sectorial and positive definite.

Chapter 7

Applications

In this section we will show the applications of the theory described earlier to some operators.

Assume that $(X, \|\cdot\|)$ is a Banach space, and let $A : D(A) \subseteq X \to X$ be a sectorial operator in X. We consider a scale of the Banach spaces $(X^\alpha)_{\alpha\in\mathbb{R}}$, generated by operator A and space X. Then we can consider operators $A|_{X^\alpha}\colon X^{\alpha+1} \subseteq X^\alpha \to X^\alpha$ for $\alpha \in \mathbb{R}$ given by

(i) if $\alpha = 0$, then an operator $A|_X\colon X^1 \subseteq X \to X$ coincides with $A\colon D(A) \subseteq \subseteq X \to X$,

(ii) if $\alpha > 0$, then X^α is a dense subset of X and $X^{\alpha+1}$ is densely contained in X^1, so we can consider an operator $A|_{X^\alpha}\colon X^{\alpha+1} \subseteq X^\alpha \to X^\alpha$ as a restriction of $A\colon D(A) \subseteq X \to X$ to the set $X^{\alpha+1}$, i.e.

$$A|_{X^\alpha}x = Ax, \quad x \in X^{\alpha+1},$$

(iii) if $\alpha < 0$, then the space X^α is a completion of the space X in the norm $\|\cdot\|_{X^\alpha}$, so we can consider an operator $A|_{X^\alpha}\colon X^{\alpha+1} \subseteq X^\alpha \to X^\alpha$ as a realization (extension) of $A\colon D(A) \subseteq X \to X$, i.e.

$$A|_{X^\alpha}x = Ax, \quad x \in X^{\alpha+1} \cap X.$$

First we prove that when an operator $A\colon D(A) \subseteq X \to X$ is densely defined and closed, then all operators $A|_{X^\alpha}\colon X^{\alpha+1} \subseteq X^\alpha \to X^\alpha$ for $\alpha \in \mathbb{R}$ are closed.

Lemma 7.0.1. *When $A|_X\colon D(A) \subseteq X \to X$ is a closed densely defined operator such that $(-\infty, 0] \subseteq \rho(A|_X)$ and there exists a constant $M \geq 1$ such that*

$$\|(s-A)^{-1}\| \leq \frac{M}{|s|}, \quad s \leq 0,$$

then all operators $A|_{X^\alpha}\colon X^{\alpha+1} \subseteq X^\alpha \to X^\alpha$ for all $\alpha \in \mathbb{R}$ are closed.

Proof. Take $\alpha > 0$ and a sequence $(x_n)_{a\in\mathbb{N}}$ of the elements of the space $X^{\alpha+1}$, which converges to some x in X^α such that $(A|_{X^\alpha}x_n)_{n\in\mathbb{N}}$ also converges to some y in X^α. Then we have that $(A|_X)^\alpha x_n \in D(A|_X)$ for all $n \in \mathbb{N}$ and $(A|_X)^\alpha x_n$ tends to $(A|_X)^\alpha x$, $A|_X(A|_X)^\alpha x_n$ tends to $(A|_X)^\alpha y$. The closeness of an operator $A|_X$ implies $x \in D(A|_X)$ and $(A|_X)^{\alpha+1}x = (A|_X)^\alpha y$ so $A|_{X^\alpha}x = y$.

For $\alpha < 0$ the proof is similar. □

Now we prove similar lemma for sectorial operators.

Lemma 7.0.2. *When $A|_X\colon X^1 \subseteq X \to X$ is a sectorial operator with* $\mathrm{Re}\sigma(A|_X) > 0$*, then also all operators $A|_{X^\alpha}\colon X^{\alpha+1} \subseteq X^\alpha \to X^\alpha$ for $\alpha \in \mathbb{R}$ are sectorial.*

Proof. From the previous lemma we know that operators $A|_{X^\alpha}$ are closed and densely defined for all $\alpha \in \mathbb{R}$. Now we estimate the resolvent of $A|_{X^\alpha}$. Since $A|_X$ is sectorial in X, there are $a \in \mathbb{R}$, $M > 0$ and $\phi \in (0, \frac{\pi}{2})$ such that sector $S_{a,\phi}$ is contained in the resolvent set $\rho(A|_X)$ of $A|_X$ and

$$\|(\lambda - A|_X)^{-1}x\|_X \leq \frac{M}{|\lambda - a|}\|x\|_X, \quad \lambda \in S_{a,\phi}, x \in X^1. \tag{7.1}$$

Let $\alpha > 0$. For $A|_{X^\alpha}$ we will show that

$$S_{a,\phi} \subseteq \rho(A|_{X^\alpha}),$$
$$\|(\lambda - A|_{X^\alpha})^{-1}y\|_{X^\alpha} \leq \frac{M}{|\lambda - a|}\|y\|_{X^\alpha}, \quad \lambda \in S_{a,\phi}, y \in X^\alpha.$$

If $\lambda \in S_{a,\phi}$, then to each $y \in X^\alpha$ corresponds a unique $x \in X^1$ satisfying $(\lambda - A|_X)x = (A|_X)^\alpha y$. Applying $(A|_X)^{-\alpha}$ to both sides of the latter equality and noting that

$$(A|_X)^{-\alpha}(\lambda - A|_X)x = (\lambda - A|_X)(A|_X)^{-\alpha}x, \quad x \in X^1 \tag{7.2}$$

we conclude that the equation $(\lambda - A|_{X^\alpha})\tilde{x} = y$ has a unique solution $\tilde{x} \in X^{1+\alpha}$ for each $y \in X^\alpha$. Consequently, the inverse $(\lambda - A|_{X^\alpha})^{-1}$ is defined on X^α. Based on (7.2) and the estimate (7.1), we finally verify that

$$\begin{aligned}\|(\lambda - A|_{X^\alpha})^{-1}y\|_{X^\alpha} &= \|(\lambda - A|_X)^{-1}y\|_{X^\alpha} = \|(A|_X)^z(\lambda - A|_X)^{-1}y\|_X = \\ &= \|(\lambda - A|_X)^{-1}(A|_X)^z y\|_X \leq \\ &\leq \frac{M}{|\lambda - a|}\|A^z y\|_X = \frac{M}{|\lambda - a|}\|y\|_{X^\alpha},\end{aligned}$$

for all $\lambda \in S_{a,\phi}$ and $y \in X^\alpha$.

For $z < 0$ we first note that the space $(X^\alpha, \|\cdot\|_{X^\alpha})$ is a completion of the space $(X, \|\cdot\|_{X^\alpha})$, so for every $y \in X^\alpha$ there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of the set X such that $\|y_n - y\|_{X^\alpha} \to 0$.
Take $\lambda \in S_{a,\phi}$. Then for every $n \in \mathbb{N}$ for $y_n \in X$ there exists a unique $x_n \in X^{1-\alpha}$ such that $(\lambda - A|_X)x_n = (A|_X)^\alpha y_n$. Note that:

$$(A|_X)^{-\alpha}(\lambda - A|_X)x = (\lambda - A|_X)(A|_X)^{-\alpha}x, \quad x \in X^{1-\alpha}.$$

Thus

$$\begin{aligned} y_n &= (A|_{X^\alpha})^{-\alpha}(A|_{X^\alpha})^\alpha y_n = (A|_X)^{-\alpha}(A|_X)^\alpha y_n = \\ &= (A|_{X^\alpha})^{-\alpha}(\lambda - A|_X)x_n = (\lambda - A|_X)(A|_X)^{-\alpha}x_n. \end{aligned}$$

Then defining $\tilde{x}_n = (A|_X)^{-\alpha}x_n \in X^1$ and noting that $X^{1-\alpha} \subseteq X^{1+\alpha}$ we see that

$$\tilde{x}_n = (\lambda - A|_{X^\alpha})^{-1}y_n, \quad n \in \mathbb{N}.$$

Since the operator $(\lambda - A|_{X^\alpha})^{-1}$ is bounded in X (because it is sectorial in X), it is also bounded in $X^{1-\alpha}$. Then the sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ converges to some x in $X^{1-\alpha}$, so it converges also in $X^{1+\alpha}$. Since the operator $(\lambda - A|_{X^\alpha})$ is closed (cf. Lemma 7.0.1), we obtain $x \in X^{1+\alpha}$ and $y = (\lambda - A|_{X^\alpha})x$.
We omit the proof of the estimate for resolvent set because it is almost the same as in the case $\alpha > 0$. □

Lemma 7.0.3. *Assume that the operator $A|_{L^p(\Omega)} \colon L^p(\Omega) \to L^p(\Omega)$ is sectorial. Then also the operator $A|_{H_0^1(\Omega)}$, the restriction of the operator $A|_{L^p(\Omega)}$ to the space $H_0^1(\Omega)$, and the operator $A|_{H^{-1}(\Omega)}$, the extension of $A|_{L^p(\Omega)}$ to the space $H^{-1}(\Omega)$, are sectorial.*

Proof. Since A is sectorial in $L^2(\Omega)$, there are $a \in \mathbb{R}$, $M > 0$ and $\phi \in (0, \frac{\pi}{2})$ such that the sector $S_{a,\phi}$ is contained in the resolvent set $\rho(A)$ of A and

$$\|(\lambda - A)^{-1}f\|_{L^2(\Omega)} \le \frac{M}{|\lambda - a|}\|f\|_{L^2(\Omega)}, \quad \lambda \in S_{a,\phi}, f \in L^2(\Omega). \tag{7.3}$$

First, we prove that A is sectorial in $H^{-1}(\Omega)$. Take $\lambda \in S_{a,\phi}$, $f \in H^{-1}(\Omega)$ and choose $f^0, f^1, \dots, f^n \in L^2(\Omega)$ such that $f = f^0 - \sum_{i=1}^{n} f^i_{x_1}$ (see

L.C. Evans, [13], Theorem 1, p.283 for characterization of the space $H^{-1}(\Omega)$).
Let $u \in H_0^1(\Omega)$ and $\lambda \in S_{a,\phi}$. Then

$$
\begin{aligned}
&\|\langle(\lambda - A)^{-1} f, u\rangle_{\langle H^{-1}(\Omega), H_0^1(\Omega)\rangle}\| = \\
&\qquad = \|(\lambda - A)^{-1} \int_\Omega \left[f^0 u + \sum_{i=1}^n f^i u_{x_i} \right] dx\| \\
&\qquad \le \int_\Omega \|(\lambda - A)^{-1}\| \|f^0\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \\
&\qquad\quad + \sum_{i=1}^n \|(\lambda - A)^{-1}\| \|f^i\|_{L^2(\Omega)} \|u_{x_i}\|_{L^2(\Omega)} \\
&\qquad \le \frac{M}{|\lambda - a|} \|f^0\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \sum_{i=1}^n \frac{M}{|\lambda - a|} \|f^i\|_{L^2(\Omega)} \|u_{x_i}\|_{L^2(\Omega)} \\
&\qquad \le \frac{M}{|\lambda - a|} \|u\|_{H_0^1(\Omega)} \left(\|f^0\|_{L^2(\Omega)} + \sum_{i=1}^n \|f^i\|_{L^2(\Omega)} \right).
\end{aligned}
$$

Taking infimum of the right side of the above inequality we obtain

$$
\|\langle(\lambda - A)^{-1} f, u\rangle_{\langle H^{-1}(\Omega), H_0^1(\Omega)\rangle}\| \le \frac{M}{|\lambda - a|} \|f\|_{H^{-1}(\Omega)} \|u\|_{H_0^1(\Omega)},
$$

which concludes this part of the proof.
Next, we show sectoriality of A in $H_0^1(\Omega)$. Let $f \in H_0^1(\Omega)$ and $\lambda \in S_{a,\phi}$. Then

$$
\begin{aligned}
\|(\lambda - A)^{-1} f\|_{H_0^1(\Omega)} &= \|(\lambda - A)^{-1} f\|_{L^2(\Omega)} + \sum_{i=1}^n \|\frac{\partial}{\partial x_i} (\lambda - A)^{-1} f\|_{L^2(\Omega)} \\
&= \|(\lambda - A)^{-1} f\|_{L^2(\Omega)} + \sum_{i=1}^n \|(\lambda - A)^{-1} \frac{\partial}{\partial x_i} f\|_{L^2(\Omega)} \le \\
&\le \frac{M}{|\lambda - a|} \|f\|_{L^2(\Omega)} + \sum_{i=1}^n \frac{M}{|\lambda - a|} \|\frac{\partial}{\partial x_i} f\|_{L^2(\Omega)} = \\
&= \frac{M}{|\lambda - a|} \left(\|f\|_{L^2(\Omega)} + \sum_{i=1}^n \|\frac{\partial}{\partial x_i} f\|_{L^2(\Omega)} \right) = \\
&= \frac{M}{|\lambda - a|} \|f\|_{H_0^1(\Omega)}
\end{aligned}
$$

□

Now we formulate a theorem which describes the relation between a complex interpolation method and scales of the Banach spaces.

Theorem 7.0.4. *Let A be a sectorial operator in a Banach space X such that* $\mathrm{Re}\sigma(A) > 0$ *and $\|A^{it}\|_{L(X)} \leq const.(\varepsilon)$ for all $t \in [-\varepsilon, \varepsilon]$ with some $\varepsilon > 0$ (such A have 'bounded imaginary power'). Then the following interpolation formula holds*

$$[X^\alpha, X^\beta]_\theta = X^{(1-\theta)\alpha+\theta\beta}, \quad \text{for } \alpha, \beta \geq 0, \theta \in (0,1),$$

together with the corresponding moments inequality

$$\|v\|_{X^{(1-\theta)\alpha+\theta\beta}} \leq c(\theta)\|v\|_{X^\beta}^\theta \|v\|_{X^\alpha}^{1-\theta}, \quad v \in X^\alpha \cap X^\beta, \alpha, \beta \geq 0, \theta \in (0,1).$$

Proof of this theorem can be found in paragraph 1.9.3 of [40].

Because $X = X^0$ and $D(A) = X^1$, taking $\alpha = 0$ and $\beta = 1$ in Theorem 7.0.4 we conclude that:

Corollary 7.0.5. *Under the assumptions of Theorem 7.0.4, X^θ are intermediate spaces between X and $D(A)$ when $\theta \in (0,1)$, that is,*

$$X^\theta = D(A^\theta) = [X, D(A)]_\theta$$

and

$$\|v\|_\theta \leq c(\theta)\|Av\|_X^\theta \|v\|_X^{1-\theta}, \quad v \in D(A).$$

Let us now come back to the Proposition 6.1.3 where the perturbation result for the sectorial operator was introduced. Having shown that the perturbed operator is sectorial it is important to observe how the perturbation influences the domains of its fractional powers. For perturbations described in Proposition 6.1.3 it is easy to conclude that the corresponding fractional power spaces remain unchanged.

Corollary 7.0.6. *Let $A\colon D(A) \to X$ be a sectorial operator in a Banach space X and consider a closed, linear operator $B\colon D(B) \to X$ such that $D(A) \subseteq D(B) \subseteq X$ and B is subordinated to A according to the condition*

$$\|Bv\|_X \leq c\|Av\|_X + c'\|v\|_X, \quad v \in X. \tag{7.4}$$

Additionally, if the operators A and $A + B$ have bounded imaginary powers and $\mathrm{Re}\sigma(A) > 0$ *and* $\mathrm{Re}\sigma(A + B) > 0$, *then the following equality holds*

$$D\left((A+B)^z\right) = D(A^z), \quad z \in (0,1).$$

Proof. Since $D(A) \subseteq D(B)$, we have $D(A+B) = D(A)$. Then using Corollary 7.0.5 we obtain

$$D(A^z) = [X, D(A)]_z = [X, D(A+B)]_z = D\left((A+B)^z\right), \quad z \in (0,1).$$

□

Following examples justify the consideration of the spaces with fractional exponents on scale. In particular, it will be showed that investigating only the classical Sobolev spaces is insufficient and we must consider also the fractional Sobolev spaces.

Example 7.0.7 (Korteweg-de Vries Equation). In this example Cauchy's problem for the Korteweg-de Vries equation

$$\begin{cases} u_t + u_{xxx} + uu_x = 0, & t \geq 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases} \tag{7.5}$$

will be discussed.

A. Pazy in his book [31] introduces the spaces $H^s(\mathbb{R})$, for every $s \in \mathbb{R}$, as follows: let $u \in L^2(\mathbb{R})$ and then we set

$$\|u\|_s = \left(\int_{\mathbb{R}} (1+\xi^2)^s |Fu(\xi)|^2 \, d\xi\right)^{\frac{1}{2}},$$

where F denotes the Fourier transform. The linear space of the functions $u \in L^2(\mathbb{R})$, for which $\|u\|_s$ is finite, is a pre-Hilbert space equipped with the scalar product

$$(u,v)_s = \int_{\mathbb{R}} (1+\xi^2)^s Fu(\xi)\overline{Fv}(\xi) \, d\xi.$$

The complection of this space with respect to the norm $\|\cdot\|_s$ is a Hilbert space denoted by $H^s(\mathbb{R})$.

Proposition 7.0.8. *Spaces $H^s(\mathbb{R})$ coincide with $W^{s,2}(\mathbb{R})$ for $s \in \mathbb{R}$.*

Proof of this fact can be found for example in book of L. Tartar [37]. Pazy's definition of the spaces $H^s(\mathbb{R})$ is one of the oldest attempt to expand the definition of classical Sobolev spaces for fractional indexes.

A. Pazy in [31], paragraph 8.5, proves the theorem on the local solvability of the KdV equation.

Theorem 7.0.9 (Theorem 5.6 in [31]). *For every $u_0 \in H^s(\mathbb{R})$, $s > 3$ there is a $T > 0$ such that the initial value problem* (7.5) *has a unique solution $u \in C([0,T], H^s(\mathbb{R})) \cap C([0,T], L^2(\mathbb{R}))$.*

The aim of the next examples is to show that to treat certain equations of mathematical physics it is **necessary** to consider the spaces X^α with fractional α.

Example 7.0.10 (Quasi-Geostrophic Equation). P. Constantin in [10] and A. Córdoba, D. Córdoba in [11] study the two dimensional quasi-geostrophic equation which has the form

$$\begin{cases} (\partial_t + u\cdot\nabla)\theta = -\kappa(-\Delta)^{\frac{\alpha}{2}}\theta, \\ u = \nabla^\perp\psi = (-\partial_{x_2}, \partial_{x_1})\psi, \quad \theta = -(-\Delta)^{\frac{1}{2}}\psi, \end{cases} \tag{7.6}$$

with the parameters $\alpha \in (0,1)$ and $\kappa > 0$. This equation is an important character of Geophysical Fluid Dynamics. Here ψ is the stream function, θ represents the potential temperature, u the velocity and κ is the viscosity. The equation will be considered with initial data $\theta(x,0) = \theta_0(x)$. The velocity u can be written in the following norm

$$u = (-\partial_{x_2}\Lambda^{-1}\theta, \partial_{x_1}\Lambda^{-1}\theta),$$

where Λ represents the operator $(-\Delta)^{\frac{1}{2}}$.

In the supercritical cases, $0 \le \alpha \le 1$, we have the following global existence results for small data.

Theorem 7.0.11. *Let $\kappa > 0$, $0 \le \alpha \le 1$, and assume that the initial data satisfies $\|\theta_0\|_{W^{m,2}} \le \frac{\kappa}{C}$ (where $m > 2$ and $C = C(m) < \infty$ is a fixed constant). Then there exists a unique solution to* (7.6) *which belongs to $W^{m,2}$ for all time $t > 0$.*

In the critical case $\alpha = 1$, $\kappa > 0$, we have the following:

Theorem 7.0.12 (Global existence for small data). *Let θ be a weak solution of* (7.6) *with an initial data $\theta_0 \in W^{\frac{3}{2},2}$ satisfying $\|\theta_0\|_{L^\infty} \le \frac{\kappa}{C}$ (where $C < \infty$ is a fixed constant). Then $\theta \in C^1([0,\infty); W^{\frac{3}{2},2})$ is a classical solution.*

Example 7.0.13. Another class of equations forcing us to consider fractional spaces are given by equations with so called *anomalous diffusion.* The examples of such equation are *multifractal conservation laws*, i.e.

$$u_t + f(u)_x = Au,$$

where $x \in \mathbb{R}$, $t \ge 0$, $u\colon \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$, $f(u)$ is a polynomially bounded nonlinear term, and

$$A = c_0\frac{\partial^2}{\partial x^2} - \sum_{j=1}^N c_j\left(-\frac{\partial^2}{\partial x^2}\right)^{\frac{\alpha_j}{2}},$$

with $c_0, c_j \geq 0$, is the diffusion operator including fractional powers of order $\frac{\alpha_j}{2}$, $0 < \alpha_j < 2$, of the square root of the second derivative with respect to x. Such problem is related to the, so called, Lévy stochastic processes. This problem is a generalization of the one-dimensional Burgers equation

$$u_t + 2uu_x = u_{xx}.$$

This type of equations was considered e.g. by P. Biler, G. Karch and W. Woyczynski in paper [6]. These authors focus their attention on simpler problem called *fractal Burgers-type equation*

$$u_t - u_{xx} + D^\alpha u + 2uu_x = 0 \tag{7.7}$$

with initial condition $u(0,x) = u_0(x)$, where $D^\alpha = (-\frac{\partial^2}{\partial x^2})^{\frac{\alpha}{2}}$ is the fractional symmetric derivative of order $\alpha \in (0,2)$.

By a solution to the Cauchy problem for the fractal Burgers equation we mean a mild solution, i.e., a function $u \in C([0,T];X)$ satisfying the Duhamel formula

$$u(t) = e^{tA} * u_0 - \int_0^t \frac{\partial}{\partial x} e^{(t-\tau)A} * u^2(\tau)\, d\tau$$

for each $t \in (0,T)$. Here X is a suitable Banach space such that e^{tA} acts as a strongly continuous semigroup in X. As usual e^{tA} denotes the (integral kernel of the) semigroup generated by the operator $A = \frac{\partial^2}{\partial x^2} - D^\alpha$, so that $v = e^{tA} * u_0$ solves the linear equation $v_t = v_{xx} - D^\alpha v$ with the initial condition $v(0) = u_0$. However, our preferred choice is $X = L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, which leads to a small modification of the above definition. Because of poor properties of e^{tA} on $L^\infty(\mathbb{R})$, we need u to belong to a larger space $\mathcal{C}([0,T];X)$ of weakly continuous functions with values in X.

Theorem 7.0.14. *Assume that* $0 < \alpha < 2$. *Given* $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, *there exists a unique mild solution* $u = u(x,t)$ *to the problem* (7.7) *in the space* $C([0,\infty); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$. *This solution satisfies the inequalities*

$$\|u(t)\|_{L^1} \leq \|u_0\|_{L^1},$$
$$\|u(t)\|_{L^2} \leq C(1+t)^{-1/(2\alpha)},$$

for all $t > 0$ *and a constant* $C > 0$.

P. Biler *et al* in their paper [6] prove also the following theorem of the large–time asymptotics for the solutions of (7.7):

Theorem 7.0.15. *Let* $0 < \alpha < 2$. *Assume that* u *is a solution to the Cauchy problem* (7.7) *with* $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. *For every* $p \in [1,\infty]$ *there exists a constant* C *such that*

$$\|u(t) - e^{tA} * u_0\|_{L^p} \le C \begin{cases} t^{-(1-1/p)/\alpha - 2/\alpha + 1}, & \text{for } 1 < \alpha < 2, \\ t^{-(1-1/p)/\alpha - 1/\alpha} \log(1+\alpha), & \text{for } \alpha = 1, \\ t^{-(1-1/p)/\alpha - 1/\alpha}, & \text{for } 0 < \alpha < 1, \end{cases}$$

for all $t > 0$.

Chapter 8

The abstract Cauchy problem

Let us consider the Cauchy problem

$$\begin{cases} u_t + Au = F(u), & t > 0, \\ u(0) = u_0, \end{cases} \tag{8.1}$$

where A is sectorial in a Banach space X. Without loss of generality (adding – if necessary – a term cu to both sides of (8.1)), we may assume that A is sectorial and $\mathrm{Re}\sigma(A) > 0$.

Definition 8.0.1. Let X be a Banach space, $z \in [0,1)$ and u_0 be an element of X^z. If, for some real $\tau > 0$, a function $u \in C([0,\tau), X^z)$ satisfies

$$\begin{aligned} &u(0) = u_0, \\ &u \in C^1((0,\tau), X), \\ &u(t) \in D(A), \quad \text{for all } t \in (0,\tau), \\ &u \text{ solves the first equation in (8.1) in } X \text{ for all } t \in (0,\tau), \end{aligned}$$

then u is called a *local X^z-solution of the problem* (8.1).

Theorem 8.0.2. *Let X be a Banach space, $A\colon D(A) \to X$ a sectorial operator in X with $\mathrm{Re}\sigma(A) > 0$. Let $F\colon X^z \to X$ be Lipschitz continuous on bounded subsets of X^z for some $z \in [0,1)$. Then, for each $u_0 \in X^z$, there exists a unique X^z-solution $u = u(t,u_0)$ of* (8.1) *defined on its maximal interval of existence $[0,\tau_{u_0})$, which means that either $\tau_{u_0} = +\infty$ or*

$$\text{if } \tau_{u_0} < +\infty \text{ then } \limsup_{t \to \tau_{u_0}^-} \|u(t,u_0)\|_{X^z} = +\infty. \tag{8.2}$$

We can find the proof of this Theorem in D. Henry [20], section 3.3. Following examples show some equations, which have local X^z-solution.

8.1 Examples and applications

Example 8.1.1. Regular spaces will be used to solve the Cauchy's problem in this example. Expected solution has to be more regular. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $\partial\Omega \in C^2$ and let us consider the Cauchy's problem

$$\begin{cases} u_t - \Delta u = 0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{8.3}$$

An abstract operator $A \colon X \supseteq D(A) \to X$ in the *base space* X will be defined by the differential operator $-\Delta$ subjected to suitable boundary conditions. Usually we take set X as $L^p(\Omega)$ and the domain

$$D(A) = W^{2,p}_{\{D\}}(\Omega) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega).$$

But we can select the base space in another way, taking

$$X^1 = D(A) = W^{2,p}_{\{D\}}(\Omega) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$$

as an example. We can define the base space as

$$X^\alpha = [X, X^1]_\alpha, \quad \alpha \in (0,1),$$

where $[\cdot,\cdot]_\alpha$ denote the complex interpolation functor. Then we can consider an operator $\tilde{A}$ in the base space $\tilde{X} = X^\alpha$ with the domain $D(\tilde{A}) = \tilde{X}^1 = X^{\alpha+1}$ as a restriction of the operator A, which means that $\tilde{A}u = Au$ for all $u \in X^{\alpha+1}$.

Using Sobolev's embedding theorem (see R.A. Adams [1], pp.97-98, Theorem 5.4) we know that $W^{k,p}(\Omega) \subseteq C^{m+\mu}(\Omega)$ if and only if $0 \le \mu \le k - m - \frac{n}{p} < 1$. We will find the conditions which allow us to select $p \in [1,\infty]$ such that the solutions of the problem (8.3) will belong to the space $C^{m+\mu}(\Omega)$.

Taking $p \ge p_0 = \frac{n}{-\mu+m-k}$ we have

$$k - m - \frac{n}{p} \ge k - m - n\frac{-\mu+m-k}{n} = \mu,$$

what shows, that p_0 is the smallest exponent which assures the embedding of the Sobolev space $W^{k,p_0}(\Omega)$ in the space of continuous functions.

In this example operator A is the Laplace operator with Dirichlet boundary conditions, so to find continuous solutions of (8.3) we have to know that the base space is contained in $C^\varepsilon(\Omega)$ for some $\varepsilon \in [0,1)$, in order to what is

satisfied if the domain $D(A)$ is contained in $C^{2+\varepsilon}(\Omega)$. Taking the base space typically as $L^p(\Omega)$ and $D(A) = W^{2,p}_{\{D\}}(\Omega) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ we have to show that $W^{2,p}(\Omega) \subseteq C^{2+\varepsilon}(\Omega)$ and $W^{1,p}_0(\Omega) \subseteq C^{2+\varepsilon}(\Omega)$. This is true for $p \geq p_0 = \frac{n}{1-\varepsilon}$.
It means that when we consider the problem (8.3) in the base space $L^p(\Omega)$ with $p \geq p_0$, then every solution u has the property that Δu is continuous on $\Omega \subseteq \mathbb{R}^n$ and so is u_t.

The theorem presented below gathers what was shown in the example.

Theorem 8.1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $\partial\Omega \in C^2$. Let us consider the Cauchy problem (8.3) in the base space $L^p(\Omega)$. Then, if $p \geq p_0 = \frac{n}{1-\varepsilon}$, every solution $u \in W^{2,p}_{\{D\}}(\Omega) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ of (8.3) has the property that u_t and Δu are continuous on $\Omega \subseteq \mathbb{R}^n$.*

Example 8.1.3. Let us consider following problem

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in } D_T = [0,T] \times \Omega, \\ u|_{\partial\Omega} = 0, \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{8.4}$$

where $T > 0$ and $\Omega \subseteq \mathbb{R}^n$ is an open bounded set with $\partial\Omega \in C^2$. We will search for the solutions u in the base space $W^{1,p}_0(D_T)$. Our aim is to set $p \in [1, \infty)$ such that all solutions are continuous, i.e. $W^{1,p}(D_T) \subseteq C^\mu(D_T)$ for some $\mu \in [0,1)$. Let's take $p \geq p_0 = \frac{n}{1-\mu}$. Then we see that $1 - \frac{n}{p} \geq$ $\geq 1 - \frac{n}{p_0} = \mu \geq 0$, and from Sobolev's embedding theorem we conclude that $W^{1,p}(D_T) \subseteq C^\mu(D_T)$.

Since $u \in W^{1,p}(D_T)$ we know that all derivatives u_t, u_{x_i} for $i \in \{1, \ldots, n\}$ belong to the space $L^p(D_T)$ and using equations (8.4) we claim that $\Delta u = u_t \in L^p(D_T)$.

Now, our purpose is to show that u belongs to $W^{2,p}(D_T)$. Knowing that $u \in W^{1,p}_0(\Omega)$ and using the Calderón-Zygmund type estimations (see Theorem 8.12 and Theorem 8.13 in D. Gilbarg, N. Trudinger [18], pp.186–187) we get that $u \in W^{2,p}(\Omega)$ and fulfills an estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C(\|\Delta u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}),$$

for some constant $C > 0$ depending on Ω, n. Then, for every multiindex α with $|\alpha| = 2$ we have $D^\alpha u \in L^p(\Omega)$, so there exist constants M_α such that

$\|D^\alpha u\|_{L^p(\Omega)} \le M_\alpha < \infty$, $(|\alpha| = 2)$. Thus we have the estimation:

$$\begin{aligned}
\|u\|_{W^{2,p}(D_T)} &= \sum_{|\alpha|=2} \|D^\alpha u\|_{L^p(D_T)} + \|u\|_{W^{1,p}(D_T)} = \\
&= \sum_{|\alpha|=2} \left(\int_0^T \int_\Omega |D^\alpha u|^p \, dx \, dt \right)^{\frac{1}{p}} + \|u\|_{W^{1,p}(D_T)} \le \\
&\le \sum_{|\alpha|=2} \left(\int_0^T M_\alpha^p \, dt \right)^{\frac{1}{p}} + \|u\|_{W^{1,p}(D_T)} = \\
&= \sum_{|\alpha|=2} M_\alpha \cdot T^{\frac{1}{p}} + \|u\|_{W^{1,p}(D_T)} < \infty.
\end{aligned}$$

This example showed that, under our assumptions and when $p > \frac{n}{1-\mu}$, every solution u of (8.4) which belongs to $W_0^{1,p}(D_T)$, is also an element of the space $C^\mu(D_T) \cap W^{2,p}(D_T)$. Consequently, the above example lead us to the theorem.

Theorem 8.1.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $\partial\Omega \in C^2$, $p \in [1, \infty)$ and $T > 0$. Assume that $u \in W_0^{1,p}(D_T)$ is a solution of the problem (8.4). Then, if $p > p_0 = \frac{n}{1-\mu}$, the function u belongs to the space $C^\mu(D_T) \cap W^{2,p}(D_T)$.*

Example 8.1.5. Previous examples showed us how to use the regular spaces to solve the Dirichlet problem. Now we will present how we can use the hyper-spaces to solve such a problem. Using Theorem 8.0.2 to get a solutions of

$$\begin{cases} u_t - \Delta u = f(u), \\ u|_{\partial\Omega} = 0, \\ u(0, x) = u_0 \in X^\gamma, \end{cases} \tag{8.5}$$

we have to show that the function $f \colon X^\gamma \to X$ satisfies a Lipschitz condition on bounded subsets of X^γ. It is often easier to verify this condition in a bigger space on the scale.

Let us take $X = L^p(\Omega)$ where $\Omega \subseteq \mathbb{R}^n$ is an open bounded set with $\partial\Omega \in C^2$, as an example. Then the Banach scale is in the form $X^\gamma = W_B^{2\gamma,p}(\Omega)$, where $W_B^{2\gamma,p}(\Omega)$ denotes the fractional Sobolev space with Dirichlet boundary conditions. If we choose the base space as $\tilde{X} = X^{-\frac{1}{2}} = W_B^{-1,p}(\Omega)$, then $f \colon X^{-\frac{1}{2}+\gamma} \to X^{-\frac{1}{2}}$.

Further, we will consider exemplary nonlinearity f given by:

$$f(u) = u|u|^s,$$

where $s \in \mathbb{R}^+$. We would like to find the largest possible value of s such that f is Lipschitz continuous on bounded subsets of $X^{-\frac{1}{2}+\gamma}$ with certain $\gamma \in [0,1]$.
Let us take $\gamma = 1$. Then we have $f\colon W^{1,p}_B(\Omega) \to W^{-1,p}(\Omega)$ and (using the mean value theorem):

$$\begin{aligned} |f(u) - f(v)| &= |u|u|^s - v|v|^s| \leq |u|\,||u|^s - |v|^s| + |u-v||v|^s \leq \\ &\leq |u|s|u-v|\left(|u|^{s-1} + |v|^{s-1}\right) + |u-v||v|^s = \\ &= |u-v|\left(s|u|^s + s|u||v|^{s-1} + |v|^s\right) \leq \\ &\leq c_s|u-v|\left(|u|^s + |v|^s\right). \end{aligned}$$

Our aim is to estimate the difference $\|f(u) - f(v)\|_{W^{-1,p}(\Omega)}$. For that purpose, going through the dual spaces, we first observe that $L^{\frac{np}{n+p}}(\Omega) \subseteq W^{-1,p}(\Omega)$. To prove this fact, we will show the corresponding embedding for dual spaces $W^{1,p'}(\Omega) \subseteq L^{r'}(\Omega)$ where p' is Hölder conjugate exponent to p ($\frac{1}{p} + \frac{1}{p'} = 1$) and r' to $\frac{np}{n+p}$. From the Sobolev's theorem (see Theorem 6 (General Sobolev inequatities) in Evans [13], page 270) we know that this embedding appears when $\frac{1}{r'} \geq \frac{1}{p'} - \frac{1}{n}$ and $1 < \frac{n}{p'}$. If we take $r' = \frac{-np}{p-np+n}$, we will have

$$\frac{1}{r'} = \frac{p-np+n}{-np} = \frac{p}{-np} + \frac{n-np}{-np} = \frac{p-1}{p} - \frac{1}{n} = \frac{1}{p'} - \frac{1}{n},$$

so that the embedding $W^{1,p'}(\Omega) \subseteq L^{r'}(\Omega)$ holds.

Let us fix a bounded set $B \subseteq W^{1,p}_B(\Omega)$ and let $u, v \in B$. Let us choose $P = \frac{p+n}{n}$, $Q = \frac{p+n}{p}$ so that $\frac{1}{P} + \frac{1}{Q} = 1$. Then using Hölder inequality, with exponents P and Q, we obtain:

$$
\begin{aligned}
\|f(u)-&f(v)\|_{W^{-1,p}(\Omega)} \leq \\
&\leq c_B \|f(u)-f(v)\|_{L^{\frac{np}{n+p}}(\Omega)} = \\
&= c_B \|u|u|^s - v|v|^s\|_{L^{\frac{np}{n+p}}(\Omega)} = \\
&= c_B \left(\int_\Omega |u|u|^s - v|v|^s|^{\frac{np}{n+p}} \, dx \right)^{\frac{n+p}{np}} \leq \\
&\leq c_B \left(\int_\Omega ||u-v| c_s \left(|u|^s + |v|^s\right)|^{\frac{np}{n+p}} \, dx \right)^{\frac{n+p}{np}} \leq \\
&\leq c_{s,B} \left(\int_\Omega |u-v|^{\frac{np}{n+p}\cdot P} \, dx \right)^{\frac{1}{P}\frac{n+p}{np}} \cdot \\
&\quad \cdot \left(\int_\Omega (2\max(|u|,|v|))^{s\cdot \frac{np}{n+p}\cdot Q} \, dx \right)^{\frac{1}{Q}\frac{n+p}{np}} .
\end{aligned}
\tag{8.6}
$$

We see that $W^{1,p}(\Omega) \subseteq L^{\frac{np}{n+p}P}(\Omega) = L^p(\Omega)$, since $p = \frac{np}{n+p}P$. Now our goal is to prove that $W^{1,p}(\Omega) \subseteq L^{s\frac{np}{n+p}Q}(\Omega) = L^{sn}(\Omega)$. If $p \geq n$, then we obtain that $W^{1,p}(\Omega) \subseteq L^\infty(\Omega) \subseteq L^{sn}(\Omega)$ for all $s \leq 0$, since $0 \leq 1 - \frac{n}{p}$ and Ω is bounded subset of $\mathbb{R}^n$. If $p < n$, we define $s_1 = \frac{p}{n-p}$ and taking $s \leq s_1$ we have that $W^{1,p}(\Omega) \subseteq L^{sn}(\Omega)$, since $-\frac{n}{sn} \leq -\frac{n}{\frac{np}{n-p}} = \frac{p-n}{p} = 1 - \frac{n}{p}$. It follows that both integrals on the right side of (8.6) converge and we obtain:

$$
\begin{aligned}
\|f(u)-f(v)\|_{W^{-1,p}(\Omega)} &\leq c_{s,B} \left(\int_\Omega |u-v|^{\frac{np}{n+p}\cdot P} \, dx \right)^{\frac{1}{P}\frac{n+p}{np}} \cdot \\
&\quad \cdot \left(\int_\Omega (2\max(|u|,|v|))^{s\cdot \frac{np}{n+p}\cdot Q} \, dx \right)^{\frac{1}{Q}\frac{n+p}{np}} \leq \\
&\leq c_{s,B} \left(\int_\Omega |u-v|^p \, dx \right)^{\frac{1}{p}} \cdot \\
&\quad \cdot \left(\int_\Omega (2\max(|u|,|v|))^p \, dx \right)^{\frac{1}{p}} = \\
&= c_{s,B} \|u-v\|_{L^p(\Omega)} M_B \leq C(s,B) \|u-v\|_{W^{1,p}(\Omega)},
\end{aligned}
$$

where M_B is a constant such that $\left(\int_\Omega (2\max(|u|,|v|))^p \, dx\right)^{\frac{1}{p}} \leq M_B$ (it exists because u and v belong to a bounded set in $W^{1,p}_B(\Omega)$).

This example showed that if we take the exponent $s \leq s_1 = \frac{p}{n-p}$ for function $f\colon W^{1,p}_B(\Omega) \to W^{-1,p}(\Omega)$, $f(u) = u|u|^s$, the Lipschitz condition on bounded sets is satisfied and we can use the Theorem 8.0.2 for solving the problem (8.5). Such s_1 is called a *critical exponent*, which means that for every $s \leq s_1$ Lipschitz condition is satisfied, but it fails for $s > s_1$. The conclusions from this example are the base for the theorem given below.

Theorem 8.1.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $\partial\Omega \in C^2$ and $X^\gamma = W_B^{2\gamma,p}(\Omega)$ be the Banach scale, where $W_B^{2\gamma,p}(\Omega)$ denotes the fractional Sobolev space with Dirichlet boundary conditions. Let us consider the problem* (8.5) *with function $f\colon W_B^{1,p}(\Omega) \to W^{-1,p}(\Omega)$, $f(u) = u|u|^s$ where $s \in \mathbb{R}^+$. Then the problem* (8.5) *has a local X^γ-solution if and only if $s \leq s_1 = \frac{p}{n-p}$.*

It is worth to observe that for $p \geq n$ the Lipschitz condition on bounded subsets of $W_B^{1,p}(\Omega)$ is satisfied for all $s \geq 0$.

Example 8.1.7. Now we consider the Cauchy's problem (8.5) with the same function f but in another base space $X = L^p(\Omega)$, where Ω is a bounded subset of $\mathbb{R}^n$ with C^2 boundary. Taking $\gamma = 1$ we notice that $X^\gamma = W_B^{2,p}(\Omega)$. So our equation has the form

$$\begin{cases} u_t - \Delta u = u|u|^s, \\ u|_{\partial\Omega} = 0, \\ u(0,x) = u_0 \in X^\gamma, \end{cases} \tag{8.7}$$

where $f\colon W_B^{2,p}(\Omega) \to L^p(\Omega)$, $f(u) = u|u|^s$. Our aim is to use Theorem 8.0.2 again to find solutions of the problem above. For the same reasons as in the previous example, when $u, v \in B \subseteq W_B^{2,p}(\Omega)$ (B is bounded in $W_B^{2,p}(\Omega)$) we get:

$$\begin{aligned} \|f(u) - f(v)\|_{L^p(\Omega)} &= \left(\int_\Omega |u|u|^s - v|v|^s|^p \, dx\right)^{\frac{1}{p}} \leq \\ &\leq c_s \left(\int_\Omega |u-v|^p \left(|u|^s + |v|^s\right)^p \, dx\right)^{\frac{1}{p}}. \end{aligned}$$

Using Hölder inequality, with exponents $\frac{n}{n-2p}$ and $\frac{n}{2p}$, we obtain

$$\|f(u)-f(v)\|_{L^p(\Omega)} \leq c_s \left(\int_\Omega |u-v|^{\frac{np}{n-2p}} \, dx\right)^{\frac{n-2p}{np}} \left(\int_\Omega \left(|u|^s + |v|^s\right)^{\frac{np}{2p}} \, dx\right)^{\frac{2p}{np}}$$

To assure that the second integral is finite we have to find the biggest s such that $W^{2,p}(\Omega) \subseteq L^{sp\frac{n}{2p}}(\Omega)$. For that purpose, if we take $2p \geq n$, we verify that $W^{2,p}(\Omega) \subseteq L^\infty(\Omega) \subseteq L^{sp\frac{n}{2p}}(\Omega)$ for each $s \geq 0$, since $0 \leq 2 - \frac{n}{2p}$ and Ω

is a bounded set. On the second hand if we take $2p < n$, $s_2 = \frac{2p}{n-2p}$ and it is enough to check for $s \le s_2$ the condition

$$-\frac{n}{spQ} \le -\frac{n}{\frac{2p}{n-2p}p\frac{n}{2p}} = -\frac{n-2p}{p} = 2 - \frac{n}{p}.$$

Then, because u, v are elements of the set B which is bounded in $W^{2,p}(\Omega)$, hence bounded in $L^{sp\frac{n}{2p}}(\Omega)$ for $s \le s_2$, we observe that the second integral $\int_\Omega (|u|^s + |v|^s)^{\frac{np}{2p}} \, dx$ is finite and bounded by some constant. Therefore our estimation has the form

$$\|f(u) - f(v)\|_{L^p(\Omega)} \le c_{s,B} \left(\int_\Omega |u - v|^{\frac{np}{n-2p}} \, dx \right)^{\frac{n-2p}{np}} = c_{s,B} \|u - v\|_{L^{\frac{np}{n-2p}}(\Omega)}.$$

At the end we notice that $W^{2,p}(\Omega) \subseteq L^{\frac{np}{n-2p}}(\Omega)$, since

$$-\frac{n}{\frac{np}{n-2p}} = -\frac{n-2p}{p} = 2 - \frac{n}{p}.$$

This finishes the proof of Lipschitz continuity of f on bounded subsets of $W^{2,p}_B(\Omega)$ because

$$\|f(u) - f(v)\|_{L^p(\Omega)} \le c_{s,B} \|u - v\|_{L^{\frac{np}{n-2p}}(\Omega)} \le \tilde{c}_{s,B} \|u - v\|_{W^{2,p}(\Omega)}.$$

The theorem presented below gathers what was shown in the example.

Theorem 8.1.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $\partial\Omega \in C^2$ and $X^\gamma = W^{2\gamma,p}_B(\Omega)$ be the Banach scale, where $W^{2\gamma,p}_B(\Omega)$ denotes the fractional Sobolev space with Dirichlet boundary conditions. Then the problem* (8.7) *has local X^γ-solution if and only if $s \le s_2 = \frac{2p}{n-2p}$.*

Finally we notice that if $2p \ge n$, the Lipschitz condition on bounded sets is fulfilled for all $s \ge 0$.

Remark 8.1.9. In previous examples we considered Cauchy's problem (8.7) with the nonlinearity $f(u) = u|u|^s$. If we assume that $2p < n$ then the critical exponent for function f with the domain $W^{1,p}(\Omega)$ equals $s_1 = \frac{p}{n}$ and in case of the domain $W^{2,p}(\Omega)$ is $s_2 = \frac{2p}{n-2p}$. It is easy to check that $s_1 < s_2$, so we can conclude that if we consider function f on smaller domain on the scale, then the critical exponent will be bigger. Consequently, f can grow faster on smaller space.

Example 8.1.10. Let us consider the n-dimensional Navier-Stokes equation

$$\begin{cases} u_t = \nu\Delta u - \nabla p - (u,\nabla)u + h, & \text{for } t>0, x\in\Omega, \\ div\ u = 0, & \\ u|_{\partial\Omega} = 0, & \\ u(0,x) = u_0, & \text{for } x\in\Omega, \end{cases} \tag{8.8}$$

where $\nu > 0$ is a constant viscosity, $u = (u_1(t,x),\dots,u_n(t,x))$ denotes velocity, $p = p(t,x)$ – pressure and $h = (h_1(t,x),\dots,h_n(t,x))$ – external force. Here Ω is a bounded domain in $\mathbb{R}^n$ with $C^{2+\varepsilon}$ boundary, with $\varepsilon \in (0,1)$.

Let us take $r \in (1,\infty)$ and introduce the space

$$X_r = \mathrm{cl}_{[L^r(\Omega)]^n}\{\phi \in [C_0^\infty(\Omega)]^n : div\ \phi = 0\}.$$

Let us denote the linear continuous projection from $[L^r(\Omega)]^n$ to X_r by $P_r : [L^r(\Omega)]^n \to X_r$, which corresponds to the Helmholtz decomposition of $[L^r(\Omega)]^n$ onto the space of divergence free vector fields and scalar function gradients such that

$$X_r = P_r[L^r(\Omega)]^n, \quad [L^r(\Omega)]^n = X_r \oplus \{\nabla\phi : \phi \in [W^{1,r}(\Omega)]^n\}.$$

Let us define an unbounded, sectorial operator A_r by

$$A_r = -\nu P_r(\Delta,\dots,\Delta).$$

which considered on the domain $D(A_r) = X_r \cap \{\phi \in [W^{2,r}(\Omega)]^n : \phi|_{\partial\Omega} = 0\}$ generates an analytic semigroup, $\mathrm{Re}\sigma(A_r) > a > 0$ and has a compact resolvent. It allow us to define fractional powers A_r^α ($\alpha \in (0,1)$) of A_r having domains $X_r^\alpha = D(A_r^\alpha)$. Further characterization of the scale $(X_r^\alpha)_{\alpha\in(0,1)}$ (see [9], pp.169–177) gives us embeddings

$$\begin{aligned} &X_r^\alpha \subseteq [W^{t,r}(\Omega)]^n \cap X_r, \quad 2\alpha \ge t, r\in(2,\infty), \\ &X_r^\alpha \subseteq [C^\mu(\overline{\Omega})]^n \cap X_r, \quad 2\alpha - \frac{n}{r} \ge k+\mu, k\in\mathbb{N}, \mu\in(0,1), r\in(2,\infty). \end{aligned}$$

Let us define $F_r = -P_r(\nabla,\dots,\nabla)u$. Then the Navier-Stokes can be studied as an abstract Cauchy's problem in X_r

$$\begin{cases} u_t + A_r u = F_r u + P_r h, & t>0, \\ u(0) = u_0. \end{cases} \tag{8.9}$$

We will show that for $\alpha \in [\frac{1}{2}, 1)$ and $r > n$ the nonlinearity $F_r \colon X_r^\alpha \to X_r$ is Lipschitz continuous on bounded sets. Let us take bounded set $U \subseteq X_r^\alpha$ and $u, v \in U$. Then the estimate (see Lemma 3.3 (iii) in Y. Giga *et al.* [17])

$$\|P_r(\phi, \nabla)\psi\|_{[L^r(\Omega)]^n} \leq c_r \|\phi\|_{[W^{1,r}|(\Omega)]^n} \|\psi\|_{[W^{1,r}|(\Omega)]^n},$$

for $\phi, \psi \in [W^{1,r}(\Omega)]^n, r > n$, implies that

$$\begin{aligned}
\|F_r u - F_r v\|_{X_r} &\leq \| - P_r(u, \nabla)u + P_r(v, \nabla)v\|_{X_r} = \\
&= \| - P_r(u, \nabla)(u - v) + P_r(u - v, \nabla)v\|_{X_r} \leq \\
&\leq c_r \|u\|_{[W^{1,r}(\Omega)]^n} \|u - v\|_{[W^{1,r}(\Omega)]^n} + \\
&\quad + c_r \|u - v\|_{[W^{1,r}(\Omega)]^n} \|v\|_{[W^{1,r}(\Omega)]^n} \leq \\
&\leq c_r \max\left(\|u\|_{[W^{1,r}(\Omega)]^n}, \|v\|_{[W^{1,r}(\Omega)]^n}\right) \|u - v\|_{[W^{1,r}(\Omega)]^n} \leq \\
&\leq c_{r,U} \|u - v\|_{[W^{1,r}(\Omega)]^n}.
\end{aligned}$$

Note that $X_r^\alpha \subseteq X_r^{\frac{1}{2}}$ for $\alpha \in [\frac{1}{2}, 1)$ and $X_r^{\frac{1}{2}}$ is continuously embedded in $X_r \cap [W^{1,r}(\Omega)]^n$. That implies Lipschitz continuous on bounded sets of X_r^α for function F_r.

The summary of this example is the theorem.

Theorem 8.1.11. *Under above assumptions the n-dimensional Navier-Stokes equation has local X_r^α-solution if $\alpha \in [\frac{1}{2}, 1)$ and $r > n$.*

Remark 8.1.12. Previous examples showed us the applications of the theorem on existence of local X^z-solutions of Cauchy's problems (8.1). It is possible to investigate these examples without abstract theory of scales of Banach spaces, using only the Sobolev embeddings. This is the proof that both presented approaches led us to the same results.

Appendix A

Theory of distributions and the Fourier transform

In this chapter the theory of distribution and Fourier transform will be reminded. Only the definitions and basic theorems will be cited, excluding the proofs, which can be found among others in the chapters I and VI of K. Yosida's monograph [41] or in paragraphs 7.58–7.61 of R.A. Adams in book [1].

A.1 Theory of distributions

Let $\Omega \subseteq \mathbb{R}^N$. Space $C_0^\infty(\Omega)$ of infinietly smooth functions with compact supports is a linear space. For any compact subset $K \subseteq \Omega$, let $\mathcal{D}_K(\Omega)$ be the set of all functions $f \in C_0^\infty(\Omega)$ such that $\operatorname{supp}(f) \subseteq K$. Define on $\mathcal{D}_K(\Omega)$ a family of seminorms by

$$p_{K,m}(f) = \sup_{\substack{|s|\leq m,\\ x\in K}} |D^s f(x)|, \quad \text{where } m < \infty.$$

Then $\mathcal{D}_K(\Omega)$ is a locally convex linear topological space and, if $K_1 \subseteq K_2$, then it follows that the topology od $\mathcal{D}_{K_1}(\Omega)$ is identical with the relative topology $\mathcal{D}_{K_1}(\Omega)$ as a subset of $\mathcal{D}_{K_2}(\Omega)$. Then the inductive limits of $\mathcal{D}_K(\Omega)$'s, where K ranges over all compact subsets of Ω, is a locally convex, linear topological space. Topologized in this way $C_0^\infty(\Omega)$ is denoted by $\mathcal{D}(\Omega)$. It is to be remarked that

$$p(f) = \sup_{x\in\Omega} |f(x)|$$

is one of the seminorms which defines the topology of $\mathcal{D}(\Omega)$.

Proposition A.1.1. *The convergence*

$$\lim_{n\to\infty} f_n = 0, \quad \textit{in } \mathcal{D}(\Omega)$$

means that the following two conditions are satisfied:

(i) *there exists a compact subset K of Ω such that supp$(f_n) \subseteq K$ for $n \in \mathbb{N}$,*
(ii) *for any differential operator D^s, the sequence $(D^s f_n)_{n\in\mathbb{N}}$ converges to 0 uniformly on K.*

Definition A.1.2. A linear functional T defined and continuous on $\mathcal{D}(\Omega)$ is called a *generalized function* or *distribution* in Ω.
The value $T(\phi)$ is called the value of the generalized function T at the *testing function* $\phi \in \mathcal{D}(\Omega)$.

A.2 The Fourier transform of rapidly decreasing functions

We start with recalling the Schwartz class $\mathcal{S}$.

Definition A.2.1. The function $f \in C_0^{\infty}(\mathbb{R}^N)$ such that

$$\sup_{x\in\mathbb{R}^N} \|x^\beta D^\alpha f(x)\| < \infty$$

for every $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $\beta = (\beta_1, \ldots, \beta_N)$ with nonnegative integers α_j, β_j, where $x^\beta = x_1^{\beta_1} \cdot x_N^{\beta_N}$ is called *rapidly decreasing* or *Schwartz test function.* The totality of such functions is denoted by $\mathcal{S}(\mathbb{R}^N)$.

Proposition A.2.2. *$\mathcal{S}(\mathbb{R}^N)$ is a locally convex linear topological space.*

Proposition A.2.3. *With respect to the topology of $\mathcal{S}(\mathbb{R}^N)$, the space $C_0^\infty(\mathbb{R}^N)$ is a dense subset of $\mathcal{S}(\mathbb{R}^N)$.*

Definition A.2.4. The *Fourier transform* Ff of function $f \in \mathcal{S}(\mathbb{R}^N)$ is defined by

$$Ff(\xi) = \frac{1}{(2\pi)^{-n/2}} \int_{\mathbb{R}^N} e^{-i\langle\xi,x\rangle} f(x)\, dx,$$

where $\xi = (\xi_1, \ldots, \xi_N)$, $x = (x_1, \ldots, x_N)$, $\langle \xi, x\rangle = \sum_{j=1}^{N} \xi_j x_j$

We also define the *inverse Fourier transform* $F^{-1}g$ of $g \in \mathcal{S}(\mathbb{R}^N)$ by

$$F^{-1}g(\xi) = \frac{1}{(2\pi)^{-n/2}} \int_{\mathbb{R}^N} e^{i\langle \xi, x\rangle} g(x)\, dx.$$

Proposition A.2.5. *The Fourier transform $f \mapsto Ff$ maps $\mathcal{S}(\mathbb{R}^N)$ linearly and continuously into $\mathcal{S}(\mathbb{R}^N)$. The inverse Fourier transform $g \mapsto F^{-1}g$ maps $\mathcal{S}(\mathbb{R}^N)$ linearly and continuously into $\mathcal{S}(\mathbb{R}^N)$.*

Theorem A.2.6 (Fourier's inverse theorem). *For every $f \in \mathcal{S}(\mathbb{R}^N)$*

$$F^{-1}Ff = f, \quad FF^{-1}f = f.$$

Remark A.2.7. From the previous theorem it is easy to see that the Fourier transform maps $\mathcal{S}(\mathbb{R}^N)$ onto $\mathcal{S}(\mathbb{R}^N)$ linearly and continuously in both directions, and the inverse Fourier transform gives the inverse mapping of the Fourier transform.

A.3 The Fourier transform of tempered distributions

Definition A.3.1. A linear functional T defined and continuous on $\mathcal{S}(\mathbb{R}^N)$ is called a *tempered distribution* (in $\mathbb{R}^N$). The totality of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^N)$

Proposition A.3.2. *Since $C_0^\infty(\mathbb{R}^N) \subseteq \mathcal{S}(\mathbb{R}^N)$ as an abstract set, and since the topology in $\mathcal{D}(\mathbb{R}^N)$ is stronger than the topology in $\mathcal{S}(\mathbb{R}^N)$, the restriction the tempered distribution to $C_0^\infty(\mathbb{R}^N)$ is a distribution in $\mathbb{R}^N$. Two different distributions define, when restricted to $C_0^\infty(\mathbb{R}^N)$, two different distributions in $\mathbb{R}^N$, because $C_0^\infty(\mathbb{R}^N)$ is dense in $\mathcal{S}(\mathbb{R}^N)$ with respect the topology of $\mathcal{S}(\mathbb{R}^N)$, and hence a distribution belongs to $\mathcal{S}'(\mathbb{R}^N)$ which vanishes on $C_0^\infty(\mathbb{R}^N)$ must vanish on $\mathcal{S}(\mathbb{R}^N)$. Therefore $\mathcal{S}'(\mathbb{R}^N) \subseteq \mathcal{D}'(\mathbb{R}^N)$.*

Definition A.3.3. Since the mapping $\phi \mapsto F\phi$ from $\mathcal{S}(\mathbb{R}^N)$ onto $\mathcal{S}(\mathbb{R}^N)$ is linear and continuous in the topology of $\mathcal{S}(\mathbb{R}^N)$, we can define the Fourier transform FT of a tempered distribution T as a tempered distribution FT defined as

$$FT(\phi) = T(F\phi), \quad \phi \in \mathcal{S}(\mathbb{R}^N).$$

Remark A.3.4. In the above sens, the Fourier transform of tempered distribution is a generalization of the Fourier transform of functions.

Theorem A.3.5 (Plancherel's Theorem). *If $f \in L^2(\mathbb{R}^N)$, then the Fourier transform FT_f of T_f is defined by a function $Ff \in L^2(\mathbb{R}^N)$, i.e.*

$$FT_f = T_{Ff}, \quad \textit{with } Ff \in L^2(\mathbb{R}^N),$$

and

$$\|Ff\| = \|f\|.$$

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Łukasz Dawidowski

**Skale przestrzeni Banacha,
teoria interpolacji wraz z zastosowaniami**

STRESZCZENIE

Celem książki jest omówienie teorii skal przestrzeni Banacha oraz teorii interpolacji wraz z podaniem przykładów ich zastosowań. Składa się z trzech części, z których dwie pierwsze opisują teorię zastosowaną następnie w trzeciej części, w której zanalizowane są przykłady jej użycia.

W pierwszej kolejności opisane są teoretyczne podstawy teorii interpolacji. Podane zostały definicje oraz podstawowe twierdzenia dotyczące konstrukcji przestrzeni interpolacyjnych (interpolacja rzeczywista i zespolona).

Druga, główna część przedstawia definicję potęg ułamkowych operatorów, w szczególności dodatnich operatorów sektorialnych. Następnie zaprezentowane jest ich zastosowanie do konstrukcji skal przestrzeni Banacha, które jako główny obiekt badań są przykładem przestrzeni interpolacyjnych. W pracy podana jest charakteryzacja skal przestrzeni Banacha, która służy jako podstawa teoretyczna do opisu zastosowań tej teorii.

W trzeciej części pokazane jest wykorzystanie podanej wcześniej teorii do badania „zachowań” operatorów na różnych poziomach skali. Udowodnione zostały twierdzenia dotyczące operatorów domkniętych oraz operatorów sektorialnych. Gwarantują one, pod pewnymi założeniami, posiadanie tych własności przez operatory rozważane na dowolnych poziomach skali. Następnie opisane są konkretne równania cząstkowe, w rozwiązywaniu których można zastosować powyższą teorię. Podane przykłady dotyczą szukania rozwiązań o większej regularności pewnych równań drugiego rzędu z warunkami brzegowymi typu Dirichleta oraz rozwiązywania nieliniowego równania Laplace'a w oparciu o teorię Henry'ego, która dotyczy równań z nieliniowością spełniającą warunek Lipschitza na podzbiorach ograniczonych.

Лукаш Давидовски

Шкала Банахових пространств. Теория интерполяции и ее применение

Резюме

Целью настоящей монографии является рассмотрение теории шкалы банаховых пространств, а также теории интерполяции наряду с приведиением примеров их использования. Работа состоит из трех частей: две первые из них описывают теорию, востребованную затем в третьей части, в которой анализируются примеры ее применения.

В первую очередь описываются теоретические основания теории интерполяции. Приводятся дефиниции, а также основные утверждения, касающиеся конструкции интерполяционново пространства (рациональная и комплексная интерполяция).

Во второй, главной части монографии, представлена дефиниция дробных степеней операторов, в особенности положительных секториальных операторов. Показано также их применение в области конструкции шкалы банаховых пространств, которые в качестве главного предмета исследований являются иллюстрацией интерполяционных пространств. Кроме того, в работе дана характеристика шкалы банаховых пространств, котороя служит теоретическим основанием для описания использования этой теории.

Третья часть проецирует представленную теорию на исследование «поведения» операторов на разных уровнях шкалы. Доказаны теоремы, касающиеся замкнутых и секториальных операторов. Они гарантируют, с определенными оговорками, наличие особенностей операторов на любых уровнях шкалы. Далее описаны конкретные частичные уравнения, при решении которых можно использовать упомянутую теорию. Приведенные примеры касаются поиска решений с большей регулярностью отдельных уравнений второго порядка с граничными условиями типа Дирихле, а также решений ислинейного уравнения Лапласа на основании теории Генри, которая касается уравнений с нелинейностью, выполняющей условие Липшица на ограниченных подсистемах.

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